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# PROCEEDINGS

OF THE

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# INDEX.

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TWENTY-THIRD SESSION, 1904–1905.

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*First Meeting, 11th November* 1904.

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CHARLES TWEEDIE, Esq., M.A., B.Sc., President, in the Chair.

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## Recherches sur l'Enveloppe des Pédales des divers points d'une Circonférence par rapport à un triangle inscrit.

Par M. Edouard Collignon,

*Inspecteur général des Ponts et Chaussées en retraite.*

### CHAPITRE I.

*Mise en équation du problème à résoudre.*

Figure 1.

I. Soit ABC un triangle inscrit dans une circonférence donnée, qui a pour centre le point O, et pour rayon R la quantité OA. Si d'un point M pris sur la circonférence on abaisse les perpendiculaires ML, MN, MR sur les côtés BC, CA, AB du triangle, les pieds L, N, R de ces perpendiculaires sont situés sur une même droite RN, à laquelle on a donné le nom de *pédale* du point M ; le point M est le point *directeur* de la pédale RN.

Cette proposition, qui est due à William Wallace, est attribuée par Poncelet, sur la foi de Servois, à Robert Simson. Poncelet, dans son ouvrage sur les *Propriétés projectives des figures*, l'a généralisée, en substituant aux projections orthogonales ML, ... du point M sur les côtés, des projections obliques faisant le même angle avec les trois côtés du triangle.

A chaque position du point directeur correspond une pédale particulière, et l'ensemble de ces pédales dessine, quand on fait mouvoir le point M, une courbe enveloppe que nous allons chercher à déterminer.

Nous définirons la position du point M par l'angle MAC $= \theta$, que la droite AM, issue du point A, fait avec l'un des côtés AC du triangle donné. Cet angle se retrouve en CBM, en NRM, et si du point A nous abaissons AP perpendiculaire sur la pédale, nous le retrouvons encore en PAB.

Appelons $\alpha$ l'angle OAB, $\beta$ l'angle OAC, que font les côtés AB, AC avec le rayon AO du cercle circonscrit. Nous aurons

$$\mathrm{BAC} = \alpha + \beta, \quad \mathrm{MAA'} = \beta - \theta, \quad \mathrm{MAB} = \alpha + \beta - \theta.$$

Nous ferons AP $= p$, distance de la pédale au sommet A.

L'angle AMA′, inscrit dans la demi-circonférence, est droit, et l'on a

$$AM = AA'\cos A'AM = 2R\cos(\beta - \theta).$$

Projetons AM sur les côtés AB, AC en abaissant les perpendiculaires MR, MN ; il vient

$$AR = AM\cos MAB = 2R\cos(\beta - \theta)\cos(\alpha + \beta - \theta),$$

$$AN = AM\cos MAC = 2R\cos(\beta - \theta)\cos\theta.$$

La distance $AP = p$ du sommet A à la pédale est la projection sur AP de AR ou de AN, et l'on a

$$(1) \qquad p = AR\cos PAR = 2R\cos\theta\cos(\beta - \theta)\cos(\alpha + \beta - \theta).$$

Cette équation représente, *en coordonnées polaires*, le lieu des points P ; elle est aussi, *en coordonnées podaires*, l'équation de l'enveloppe de la pédale, perpendiculaire au rayon $p$. Dans ce dernier système de coordonnées, la distance $P\mu$ du pied P du rayon vecteur AP au point $\mu$ où la pédale touche son enveloppe est donnée par la relation

$$(2) \qquad P\mu = \frac{dp}{d\theta}.$$

Pour passer de là aux coordonnées rectangulaires rapportées à deux axes AX, AY, projetons sur ces axes le contour $AP\mu$

Il viendra

$$(3) \qquad \begin{cases} x = p\cos\theta - \dfrac{dp}{d\theta}\sin\theta \\ y = p\sin\theta + \dfrac{dp}{d\theta}\cos\theta \end{cases}$$

équations qui, prises simultanément, représentent l'enveloppe cherchée. On aurait l'équation cartésienne de la courbe en éliminant $\theta$ entre les deux équations (3).

A ces relations on peut joindre l'équation

$$(4) \qquad \rho = p + \frac{d^2p}{d\theta^2}$$

qui fait connaître le rayon de courbure. Le problème revient à former les dérivées de $p$ par rapport à l'angle $\theta$.

II. *Formation des dérivées et construction de la courbe.*

Posons, pour simplifier l'écriture,

$$(5) \qquad u = \cos\theta\cos(\beta - \theta)\cos(\alpha + \beta - \theta)$$
$$= \cos\theta\cos\phi\cos\psi,$$

en posant $p = 2\mathrm{R}u$, $\phi = \beta - \theta$, $\psi = \alpha + \beta - \theta$.

Il résulte de ces deux dernières relations que l'on a

$$\frac{d\phi}{d\theta} = \frac{d\psi}{d\theta} = -1.$$

La première dérivation donne

$$(6) \quad \frac{du}{d\theta} = -\sin\theta\cos\phi\cos\psi + \cos\theta\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi$$
$$= -\sin\theta\cos(\beta - \theta)\cos(\alpha + \beta - \theta) + \cos\theta\sin(\beta - \theta)\cos(\alpha + \beta - \theta)$$
$$+ \cos\theta\cos(\beta - \theta)\sin(\alpha + \beta - \theta)$$
$$= \cos\theta[\sin(\beta - \theta)\cos(\alpha + \beta - \theta) + \cos(\beta - \theta)\sin(\alpha + \beta - \theta)]$$
$$- \sin\theta\cos(\beta - \theta)\cos(\alpha + \beta - \theta)$$
$$= \cos\theta\sin(\alpha + 2\beta - 2\theta) - \sin\theta\cos(\beta - \theta)\cos(\alpha + \beta - \theta).$$

Les seconds membres des équations (5) et (6) peuvent se développer en substituant des sommes de lignes trigonométriques aux produits indiqués. On a identiquement

$$\cos\phi\cos\psi = \tfrac{1}{2}[\cos(\phi + \psi) + \cos(\phi - \psi)]$$

et par conséquent

$$u = \cos\theta\cos\phi\cos\psi$$
$$= \tfrac{1}{2}[\cos\theta\cos(\phi + \psi) + \cos\theta\cos(\phi - \psi)]$$
$$= \tfrac{1}{4}[\cos(\theta + \phi + \psi) + \cos(\theta + \phi - \psi) + \cos(\theta - \phi + \psi) + \cos(\theta - \phi - \psi)]$$
$$= \tfrac{1}{4}[\cos(\alpha + 2\beta - \theta) + \cos(\alpha - \theta) + \cos(\alpha + \theta) + \cos(\alpha + 2\beta - 3\theta)].$$

On en déduit, en différentiant,

$$\frac{du}{d\theta} = \tfrac{1}{4}[\sin(\alpha + 2\beta - \theta) + \sin(\alpha - \theta) - \sin(\alpha + \theta) + 3\sin(\alpha + 2\beta - 3\theta)]$$

et sous cette forme la seconde dérivation s'opère immédiatement, et conduit à un résultat simple. On a en effet

$$(7) \quad \frac{d^2u}{d\theta^2} = -\tfrac{1}{4}[\cos(\alpha + 2\beta - \theta) + \cos(\alpha - \theta) + \cos(\alpha + \theta)$$
$$+ \cos(\alpha + 2\beta - 3\theta)] - 2\cos(\alpha + 2\beta - 3\theta)\,;$$

car le terme final de la première dérivée

$$+\tfrac{3}{4}\sin(\alpha+2\beta-3\theta)$$

donne par la dérivation

$$-\tfrac{9}{4}\cos(\alpha+2\beta-3\theta)=-(2+\tfrac{1}{4})\cos(\alpha+2\beta-3\theta).$$

La parenthèse du second membre de l'équation (7) reproduit la valeur de $u$ changée de signe. On a donc

$$\frac{d^2u}{d\theta^2}+u=-2\cos(\alpha+2\beta-3\theta),$$

et en multipliant les deux membres par 2R, on trouve pour la valeur du rayon de courbure

$$\begin{aligned}(8)\qquad \rho&=p+\frac{d^2p}{d\theta^2}\\ &=2\mathrm{R}\left(u+\frac{d^2u}{d\theta^2}\right)\\ &=-4\mathrm{R}\cos(\alpha+2\beta-3\theta).\end{aligned}$$

Les valeurs de $u$ et de $\dfrac{du}{d\theta}$, multipliées de même par 2R, font connaître les expressions de $x$ et de $y$; il vient

$$\begin{aligned}(9)\qquad x&=2\mathrm{R}\left(u\cos\theta-\frac{du}{d\theta}\sin\theta\right)\\ &=2\mathrm{R}\cos(\beta-\theta)\cos(\alpha+\beta-\theta)-2\mathrm{R}\sin\theta\cos\theta\sin(\alpha+2\beta-2\theta)\\ &=\frac{\mathrm{R}}{2}[\cos(\alpha+2\beta)+2\cos\alpha+2\cos(\alpha+2\beta-2\theta)\\ &\qquad-\cos(\alpha+2\beta-4\theta)];\end{aligned}$$

$$\begin{aligned}(10)\qquad y&=2\mathrm{R}\left(u\sin\theta+\frac{du}{d\theta}\cos\theta\right)\\ &=2\mathrm{R}\cos^2\theta\sin(\alpha+2\beta-2\theta)\\ &=\frac{\mathrm{R}}{2}[\sin(\alpha+2\beta)+2\sin(\alpha+2\beta-2\theta)+\sin(\alpha+2\beta-4\theta)].\end{aligned}$$

Suivant les cas, il y aura lieu d'adopter l'une ou l'autre de ces formes.

1°. Reprenons les expressions de $x$ et de $y$:

$$(9)\qquad x=2\mathrm{R}\cos(\beta-\theta)\cos(\alpha+\beta-\theta)-2\mathrm{R}\sin\theta\cos\theta\sin(\alpha+2\beta-2\theta),$$

$$(10)\qquad y=2\mathrm{R}\cos^2\theta\sin(\alpha+2\beta-2\theta).$$

FIGURE 2.

Nous remarquerons que le premier terme de la valeur de $x$ est l'abscisse $AR = x'$ du pied R de la pédale sur le côté AB pris pour axe des abscisses. Si nous projetons le point $\mu$ en $\mu'$ sur ce même axe, nous aurons

$$\mu' R = x' - x = 2R\sin\theta\cos\theta\sin(\alpha + 2\beta - 2\theta),$$
$$\mu'\mu = y \quad = 2R\cos^2\theta\sin(\alpha + 2\beta - 2\theta).$$

Elevons au carré, et ajoutons ; nous aurons

$$\overline{R\mu}^2 = 4R^2(\cos^4\theta + \sin^2\theta\cos^2\theta)\sin^2(\alpha + 2\beta - 2\theta)$$
$$= 4R^2\cos^2\theta\sin^2(\alpha + 2\beta - 2\theta)$$

et, en extrayant la racine,

$$(11) \qquad R\mu = 2R\cos\theta\sin(\alpha + 2\beta - 2\theta),$$

quantité facile à construire.

L'angle MOA' est égal à $2OAM = 2(\beta - \theta)$, et si l'on prolonge le rayon MO jusqu' en $m$, l'angle OHB, extérieur au triangle AOH, est la somme des deux angles intérieurs $\alpha$ et $2(\beta - \theta)$ ; c'est à dire, est égal à $\alpha + 2\beta - 2\theta$. Si donc on achève le triangle $Mmm'$, dans lequel l'angle en $m'$ est droit, puisque $Mm$ est un diamètre, on aura

$$Mm' = Mm\sin(\alpha + 2\beta - 2\theta) = 2R\sin(\alpha + 2\beta - 2\theta)$$

et $\qquad R\mu = Mm'\cos\theta.$

Il suffit donc de faire glisser, de la quantité $m'R$, la corde $m'M$ pour amener le point M en un point $m''$, qui, projeté sur la pédale RN, fera connaître le point de contact $\mu$.

FIGURE 3.

Appliquée au cas particulier où le point directeur M coïncide avec le sommet C, la pédale se confond avec la hauteur CR' et la construction se résume dans le prolongement de CR' en R'', et dans le transport du segment R'R'' en $C\mu''$ ; et il n'y a pas à projeter, puisqu'ici l'angle $\theta$ est nul.

Appliquée au point M placé en A' à l'extrémité du diamètre AOA', la construction revient à projeter le point A' sur la pédale, qui est alors le côté BC lui-même, ce qui donne en $m$ la position correspondante du point de contact ; les trois droites A'C, A'B, A'$m$ sont les perpendiculaires abaissées de A' sur les trois côtés, et la dernière a pour pied le point où CB touche la courbe enveloppe.

Passons à la construction du rayon de courbure $\rho$.

La valeur de ce rayon de courbure peut se déduire de l'expression de l'ordonnée, en la différentiant. On a en effet

$$ds = \rho d\theta$$

et l'arc $ds$, projeté sur l'axe AY, donne la différentielle $dy$.
Or l'angle de $ds$ avec AY est égal à l'angle $\theta$ de AP avec AX, et l'on a

$$ds\cos\theta = dy = \rho\cos\theta\, d\theta$$

d'où l'on déduit

$$\rho = \frac{dy}{\cos\theta\, d\theta}\,.$$

Mais $y = 2\mathrm{R}\cos^2\theta\sin(\alpha + 2\beta - 2\theta)$
et par conséquent

$$\begin{aligned}\frac{dy}{d\theta} &= -4\mathrm{R}\cos\theta\sin\theta\sin(\alpha + 2\beta - 2\theta) - 4\mathrm{R}\cos^2\theta\cos(\alpha + 2\beta - 2\theta)\\ &= -4\mathrm{R}\cos\theta[\sin(\alpha + 2\beta - 2\theta)\sin\theta + \cos(\alpha + 2\beta - 2\theta)\cos\theta]\\ &= -4\mathrm{R}\cos\theta\cos(\alpha + 2\beta - 3\theta),\end{aligned}$$

$$(12)\quad \rho = \frac{dy}{\cos\theta\, d\theta} = -4\mathrm{R}\cos(\alpha + 2\beta - 3\theta),$$

c'est à dire notre équation (8).

Le signe de la formule définit, comme à l'ordinaire, le sens dans lequel la valeur absolue de $\rho$ doit être portée sur la normale ; le rayon $\rho$ doit, en définitive, être toujours porté du côté de la concavité de la courbe. Nous nous occuperons ici exclusivement de sa valeur absolue.

FIGURE 4.

Soit MAC l'angle $\theta$. Prenons le milieu I de l'arc OM et portons l'arc MI′ = MI. Si nous joignons au centre O le point I′, l'angle MOI′ sera égal à $\theta$, et l'angle MOA′ étant égal à $2(\beta - \theta)$, l'angle I′OA′ sera égal à $2\beta - 3\theta$.

L'angle OKB, extérieur au triangle OAK, sera égal à
$\alpha + \mathrm{AOI}_1' = \alpha + 2\beta - 3\theta$,
et par conséquent, pour obtenir le produit $2\mathrm{R}\cos(\alpha + 2\beta - 3\theta)$,
il suffit de mener par $\mathrm{I}_1'$ la droite $\mathrm{I}_1'\mathrm{I}_2'$ parallèle à AB,
et de joindre $\mathrm{I}'\mathrm{I}_2'$.

L'angle $I_1'I_2'I'$ sera droit, et l'on aura

$$I_1'I_2' = 2R\cos(\alpha + 2\beta - 3\theta).$$

Donc on a, en valeur absolue, $\rho = 2I_1'I_2'$.

En résumé la corde $Mm'$ conduit à la détermination du point de contact $\mu$, et la corde $I_1'I_2'$, doublée, donne la valeur absolue du rayon de courbure au point correspondant au point M.

III. *Détermination directe du point de contact $\mu$.*

Figure 5.

Soit RN la pédale du point M ; R'N' la pédale du point M' infiniment voisin ; $\mu$ le point où se coupent ces deux pédales consécutives.

Appliquons au triangle ARN, coupé par la droite R'N' le théorème des transversales. Nous aurons

$$\frac{AN'}{N'N} \times \frac{N\mu}{\mu R} \times \frac{RR'}{AR'} = 1.$$

Dans cette égalité NN' est la différentielle du segment AN prise avec le signe $-$ ; car $AN' = AN - NN'$. De même RR' est la différentielle du segment AR prise avec son signe. On a donc

$$\frac{AN + dAN}{-dAN} \times \frac{N\mu}{\mu R} \times \frac{dAR}{AR + dAR} = 1.$$

Les infiniment petits $dAN$, $dAR$ sont négligeables devant les quantités finies AN, AR, et il vient à la limite, en résolvant par rapport à $\frac{\mu R}{N\mu}$,

$$\frac{\mu R}{N\mu} = -\frac{dAR}{AR} \times \frac{AN}{dAN} = -\frac{\left(\frac{dAR}{AR}\right)}{\left(\frac{dAN}{AN}\right)},$$

de sorte que le rapport des distances du point $\mu$ aux extrémités R et N de la pédale, est égal au rapport changé de signe des dérivées logarithmiques des côtés AR, AN adjacents à ces distances.

Nous avons trouvé plus haut

$$AR = 2R\cos(\beta - \theta)\cos(\alpha + \beta - \theta),$$

$$AN = 2R\cos(\beta - \theta)\cos\theta\ ;$$

on en déduit, en prenant les différentielles logarithmiques des deux membres,

$$\frac{d\mathrm{AR}}{\mathrm{AR}}=\frac{\sin(\beta-\theta)d\theta}{\cos(\beta-\theta)}+\frac{\sin(\alpha+\beta-\theta)d\theta}{\cos(\alpha+\beta-\theta)},$$

$$\frac{d\mathrm{AN}}{\mathrm{AN}}=\frac{\sin(\beta-\theta)d\theta}{\cos(\beta-\theta)}-\frac{\sin\theta d\theta}{\cos\theta},$$

et par conséquent

$$\frac{\mu\mathrm{R}}{\mathrm{N}\mu}=-\frac{\dfrac{\sin(\beta-\theta)}{\cos(\beta-\theta)}+\dfrac{\sin(\alpha+\beta-\theta)}{\cos(\alpha+\beta-\theta)}}{\dfrac{\sin(\beta-\theta)}{\cos(\beta-\theta)}-\dfrac{\sin\theta}{\cos\theta}}=-\frac{\cos\theta\sin(\alpha+2\beta-2\theta)}{\cos(\alpha+\beta-\theta)\sin(\beta-2\theta)};$$

le facteur $\cos(\beta-\theta)$ disparaît aux deux termes de la fraction.

Si l'on multiplie haut et bas par 2R, on retrouve au numérateur la valeur de $\mu\mathrm{R}$ déterminée plus haut

$$\mu\mathrm{R}=2\mathrm{R}\cos\theta\sin(\alpha+2\beta-2\theta);$$

par suite le dénominateur donne l'égalité

$$\mathrm{N}\mu=-2\mathrm{R}\cos(\alpha+\beta-\theta)\sin(\beta-2\theta).$$

Cette seconde formule est l'application pure et simple de la première, lorsqu'on permute ensemble les côtés AB et AC, ainsi que les angles et les segments correspondants.

Pour obtenir le point $\mu$, nous avons fait usage de la perpendiculaire MR, abaissée du point M sur le côté AB du triangle; la construction nous a conduit à un point $m$ qui se projette sur la pédale au point $\mu$. On aurait pu opérer de même sur les perpendiculaires MN ou ML, abaissées sur les autres côtés; on aurait trouvé des points $n$, $l$, qui se seraient projetés sur la pédale au même point $\mu$. Les trois points $m$, $n$, $l$ sont donc en ligne droite, sur une perpendiculaire à la pédale, et l'on a ce théorème:

FIGURE 6.

Soit ABC un triangle inscrit dans le cercle O; M un point pris arbitrairement sur la circonférence. On abaisse les perpendiculaires de ce point sur les trois côtés MN, ML, MR; on prend ensuite sur ces perpendiculaires les segments

$$\mathrm{M}l=\mathrm{LL}'',\quad \mathrm{M}m=\mathrm{RM}'',\quad \mathrm{M}n=\mathrm{NN}''.$$

*Les trois points* l, m, n *seront sur une même droite perpendiculaire à la pédale* RLN, *et la coupant au point de contact* $\mu$ *de la pédale avec son enveloppe.*

## CHAPITRE II.

### *Discussion de la Courbe.*

FIGURE 7.

I. La somme $\alpha + 2\beta$, qui figure dans nos formules, se ramène aux angles du triangle donné. Soient $\alpha$, $\beta$, $\gamma$ les angles formés par les côtés du triangle avec les rayons du cercle circonscrit menés aux sommets. Nous aurons

$$\alpha+\beta=\mathrm{A},\quad \beta+\gamma=\mathrm{C},\quad \gamma+\alpha=\mathrm{B}\,;$$

d'où l'on déduit

$$\alpha=\frac{\mathrm{A}+\mathrm{B}-\mathrm{C}}{2},\quad \beta=\frac{\mathrm{A}+\mathrm{C}-\mathrm{B}}{2},\quad \gamma=\frac{\mathrm{B}+\mathrm{C}-\mathrm{A}}{2}$$

et par suite

$$\alpha+2\beta=\frac{3\mathrm{A}+\mathrm{C}-\mathrm{B}}{2}.$$

L'équation (12) devient, quand on y substitue cette valeur de $\alpha+2\beta$

$$\rho=-4\mathrm{R}\cos\left(\frac{3\mathrm{A}+\mathrm{C}-\mathrm{B}}{2}-3\theta\right). \tag{13}$$

Le rayon de courbure $\rho$, pris en valeur absolue, varie entre les limites 0 et 4R. Examinons les deux limites extrêmes.

1°. Le rayon de courbure est nul lorsque l'on a

$$\frac{3\mathrm{A}+\mathrm{C}-\mathrm{B}}{2}-3\theta=\frac{\pi}{2},\ \text{ou}\ \frac{3\pi}{2},\ \text{ou}\ \frac{5\pi}{2},$$

ce qui donne pour $\theta$ trois valeurs successives :

$$\begin{aligned}
\theta_1&=\frac{\mathrm{A}}{2}+\frac{\mathrm{C}-\mathrm{B}}{6}-\frac{\pi}{6} && \\
& && \text{diff. } \frac{\pi}{3},\\
\theta_2&=\frac{\mathrm{A}}{2}+\frac{\mathrm{C}-\mathrm{B}}{6}-\frac{\pi}{2} && \\
& && \text{diff. } \frac{\pi}{3},\\
\theta_3&=\frac{\mathrm{A}}{2}+\frac{\mathrm{C}-\mathrm{B}}{6}-\frac{5\pi}{6}, &&
\end{aligned}$$

dont les différences sont égales à $\frac{\pi}{3}$. Les directions correspondantes, issues du sommet A, coupent la circonférence aux sommets d'un triangle équilatéral.

Nous pouvons remplacer dans la première équation la demi-circonférence $\pi$ par la somme $A+B+C$ des trois angles. Il viendra

$$\theta_1 = \frac{A}{2} + \frac{C-B}{6} - \frac{A+B+C}{6} = \frac{A-B}{3}.$$

FIGURE 8.

Par le sommet C menons la corde CC′ parallèle au côté opposé BA ; cette droite intercepte sur la circonférence un arc CC′ qui correspond à un angle au centre égal à $2(B-A)$. Si donc on prend $CM_1 = \frac{1}{3}CC'$ sur la circonférence, on aura pour l'angle inscrit $CAM_1$ la valeur $\frac{B-A}{3}$, et en tenant compte du signe, on retrouve l'équation

$$\theta_1 = \frac{A-B}{3}.$$

Le point $M_1$ est donc l'un des sommets du triangle équilatéral pour lequel on aura $\rho = 0$. Les autres sommets s'obtiendront en achevant le triangle équilatéral, ou en opérant pour les sommets A et B comme nous venons de le faire pour le sommet C.

2°. On aura $\rho = 4R$, valeur maximum du rayon de courbure, en posant

$$\frac{3A+C-B}{2} - 3\theta = 0, \text{ ou } \pi, \text{ ou } 2\pi,$$

ce qui définit trois directions distinctes

$$\begin{aligned} \theta_1' &= \frac{A}{2} + \frac{C-B}{6} \\ & \qquad\qquad\qquad \text{diff. } \frac{\pi}{3}, \\ \theta_2' &= \frac{A}{2} + \frac{C-B}{6} - \frac{\pi}{3} \\ & \qquad\qquad\qquad \text{diff. } \frac{\pi}{3}, \\ \theta_3' &= \frac{A}{2} + \frac{C-B}{6} - \frac{2\pi}{3} ; \end{aligned}$$

elles sont les bissectrices des angles formés par les rayons $\theta_1$, $\theta_2$, $\theta_3$, et déterminent sur la circonférence un second triangle équilatéral, dont les sommets sont au milieu des arcs soustendus par les côtés du premier.

Les triangles équilatéraux $M_1M_2M_3$ et $M_1'M_2'M_3'$, aux sommets desquels on a, soit $\rho = 0$, soit $\rho = 4R$, sont les triangles-limites vers lesquels tendent les triangles successifs que l'on déduit du triangle ABC, en prenant pour sommets les milieux des arcs soustendus par ses côtés, et en répétant indéfiniment la même opération sur chacun

des nouveaux triangles ainsi obtenus. Le triangle $M_1M_2M_3$ est le triangle équilatéral qui diffère le moins, comme position, du triangle donné. On trouvera une étude détaillée de cette question géométrique dans le Recueil des Mémoires de l'Association française pour l'avancement des sciences, Congrès d'Oran, 1888 : *Sur certaines séries numériques.*

L'application de la méthode graphique de recherche du rayon de courbure aux points $M_1$ et $M_1'$ conduit à tracer pour $M_1$ le diamètre du cercle perpendiculaire à AB, ce qui annule la corde $I_1'I_2'$ et montre que $\rho = 0$ ; pour $M_1'$ on serait amené à tracer le diamètre parallèle à AB ; c'est ce diamètre qui devient la corde $I_1'I_2'$ et l'équation générale $\rho = 2I_1'I_2'$ équivaut alors à $\rho = 4R$.

Cherchons encore les valeurs du rayon de courbure, lorsque le point directeur M coïncide avec un des sommets du triangle, puis lorsqu'il coïncide avec l'un des points diamétralement opposés à ces sommets sur la circonférence circonscrite.

FIGURE 9.

1°. Au sommet C on a $\theta = 0$ et

$$\rho = -4R\cos(\alpha + 2\beta).$$

Le point I' coïncide alors avec le point C et le diamètre $I'I_1'$ devient ici $OI_1'$, qui fait l'angle $\alpha + 2\beta$ avec le côté AB. Le rayon de courbure est le double de la corde $I_1'I_2'$, parallèle à AB, et l'on a en valeur absolue $\rho = 2I_1'I_2'$.

Ce rayon de courbure est applicable à l'enveloppe au point $\mu$, qui est situé sur la hauteur CR, prolongée de la quantité $C\mu = I_2'R$.

FIGURE 10.

2°. Au point A', diamétralement opposé au sommet A, on a $\theta = \beta$ et l'équation donne

$$\rho = -4R\cos(\alpha - \beta).$$

La pédale correspondante est le côté CB, et le point de contact est au pied $\mu$ de la perpendiculaire abaissée du point A' sur ce côté. Soit I le milieu de l'arc CA' ; prenons $A'I_1 = A'I$ ; menons le diamètre $I_1I_1'$, et par le point $I_1'$ menons la corde $I_1'I_2'$ parallèle à AB ; nous aurons en valeur absolue

$$\rho = 2I_1'I_2',$$

ce qui vérifie la formule.

II. *Les trois points de rebroussement $\mu_1$, $\mu_2$, $\mu_3$ de la courbe enveloppe sont les sommets d'un triangle équilatéral.*

Nous avons obtenu plus haut les deux équations générales

$$(9)\quad x=\frac{R}{2}[\cos(\alpha+2\beta)+2\cos\alpha+2\cos(\alpha+2\beta-2\theta)-\cos(\alpha+2\beta-4\theta)],$$

$$(10)\quad y=\frac{R}{2}[\sin(\alpha+2\beta)+2\sin(\alpha+2\beta-2\theta)+\sin(\alpha+2\beta-4\theta)];$$

substituons à $\theta$ les valeurs $\theta_1$, $\theta_2$, $\theta_3$ qui annulent le rayon de courbure. Il viendra pour les coordonnées des points $\mu_1$, $\mu_2$, $\mu_3$, savoir

pour $\theta=\theta_1=\frac{\alpha+2\beta}{3}-\frac{\pi}{6}$, point $\mu_1$:

$$x_1=\frac{R}{2}[\cos(\alpha+2\beta)+2\cos\alpha-3\sin\theta_1],$$

$$y_1=\frac{R}{2}[\sin(\alpha+2\beta)+3\cos\theta_1];$$

pour $\theta=\theta_2=\frac{\alpha+2\beta}{3}-\frac{\pi}{2}$, point $\mu_2$:

$$x_2=\frac{R}{2}[\cos(\alpha+2\beta)+2\cos\alpha+3\sin\theta_2],$$

$$y_2=\frac{R}{2}[\sin(\alpha+2\beta)-3\cos\theta_2];$$

pour $\theta=\theta_3=\frac{\alpha+2\beta}{3}-\frac{5\pi}{6}$, point $\mu_3$:

$$x_3=\frac{R}{2}[\cos(\alpha+2\beta)+2\cos\alpha-3\sin\theta_3],$$

$$y_3=\frac{R}{2}[\sin(\alpha+2\beta)+3\cos\theta_3].$$

Des coordonnées des trois points nous pouvons déduire les valeurs de leurs distances mutuelles

$$l_{1,2}=\mu_1\mu_2,\quad l_{2,3}=\mu_2\mu_3,\quad l_{3,1}=\mu_3\mu_1.$$

Il vient, en effectuant les opérations, toutes réductions faites,

$$l_{1,2}^2 = \frac{9R^2}{2}[1 + \cos(\theta_1 - \theta_2)],$$

$$l_{2,3}^2 = \frac{9R^2}{2}[1 + \cos(\theta_2 - \theta_3)],$$

$$l_{3,1}^2 = \frac{9R^2}{2}[1 + \cos(\theta_3 - \theta_1)].$$

Les différences $\theta_1 - \theta_2$, $\theta_2 - \theta_3$, $\theta_3 - \theta_1$ sont égales à $\frac{\pi}{3}$, dont le cosinus est égal à $\frac{1}{2}$. On constate donc l'égalité des côtés du triangle $\mu_1\mu_2\mu_3$, qui ont pour longueur commune la valeur

$$l = \frac{3R\sqrt{3}}{2}.$$

La quantité $R\sqrt{3}$ est la valeur du côté du triangle équilatéral inscrit dans le cercle qui a le rayon R; le nouveau triangle équilatéral est inscrit dans un cercle de rayon $R \times \frac{3}{2}$.

Soient $\xi$ et $\eta$ les coordonnées du centre du cercle passant par les points $\mu_1$, $\mu_2$, $\mu_3$; ce point sera le centre des moyennes distances des trois sommets, et l'on aura par conséquent

$$\begin{aligned} \xi &= \frac{x_1 + x_2 + x_3}{3} \\ &= \frac{R}{2}[\cos(\alpha + 2\beta) + 2\cos\alpha - (\sin\theta_1 - \sin\theta_2 + \sin\theta_3)], \\ \eta &= \frac{y_1 + y_2 + y_3}{3} \\ &= \frac{R}{2}[\sin(\alpha + 2\beta) + (\cos\theta_1 - \cos\theta_2 + \cos\theta_3)]. \end{aligned}$$

Mais comme on a $\theta_1 = \theta_2 + \frac{\pi}{3}$, $\theta_3 = \theta_2 - \frac{\pi}{3}$, les trois sommes algébriques de sinus et de cosinus se réduisent à zéro, et l'on a simplement

$$\xi = \frac{R}{2}[\cos(\alpha + 2\beta) + 2\cos\alpha] = R\cos\alpha + \tfrac{1}{2}R\cos(\alpha + 2\beta),$$

$$\eta = \tfrac{1}{2}R\sin(\alpha + 2\beta),$$

pour les coordonnées du centre O′ du cercle circonscrit au triangle $\mu_1\mu_2\mu_3$.

FIGURE 11.

Pour construire le point O′, centre défini par ces coordonnées $\xi$ et $\eta$, prenons le milieu D du rayon OC, et projetons-le en D′ sur le côté AB. Nous aurons

$$\begin{aligned} AD' &= AO\cos\alpha + OD\cos DGB \\ &= R\,\cos\alpha + \tfrac{1}{2}R\cos(\alpha + 2\beta) = \xi, \end{aligned}$$

de sorte que le point O′ appartient à l'ordonnée DD′. Nous avons opéré sur le rayon OC du cercle donné. On pourrait de même opérer sur les rayons OA, OB, et on aurait obtenu deux droites perpendiculaires aux côtés BC, CA passant également par le point O′. Ce point O′ est donc le point de concours des hauteurs du triangle qui aurait ses sommets aux points D, E, F, milieux des rayons OC, OA, OB. Ce triangle est homothétique au triangle donné, et les points de concours des hauteurs étant des points homologues, le point O′ est le milieu de la droite OH, qui joint le centre O du cercle circonscrit au point H commun aux trois hauteurs du triangle ABC.

FIGURE 12.

Le point O′ est également éloigné des pieds

$$\text{I et K, I}' \text{ et K}', \text{ I}'' \text{ et K}''$$

des perpendiculaires abaissées, sur les côtés AB, BC, CA, des points O et H. C'est donc le centre du *cercle des neuf points*, c'est à dire du cercle qui passe par les milieux I, I′, I″ des trois côtés, par les pieds des trois hauteurs K, K′, K″, et aussi par les milieux des droites AH, BH, CH qui joignent les sommets au point de concours des hauteurs.

Les mêmes résultats s'appliquent aux trois points $\mu_1'$, $\mu_2'$, $\mu_3'$, où le rayon de courbure de l'enveloppe a pour valeur absolue 4R, son maximum. Ces trois points sont les sommets d'un triangle équilatéral, dont le côté sera égal à

$$l' = \frac{R\sqrt{3}}{2}$$

et qui a pour centre le point O′. Le rayon du cercle circonscrit à ce triangle est $\frac{R}{2}$, tiers du rayon $\frac{3R}{2}$ du cercle qui contient les trois points de rebroussement $\mu_1$, $\mu_2$, $\mu_3$.

Pour passer du point directeur $M_1$, pris sur le cercle circonscrit au triangle, au point $\mu_1$ de l'enveloppe, on pourra donc augmenter le rayon R du cercle circonscrit de la moitié de sa valeur et le porter à $\frac{3R}{2}$ ; puis déplacer le centre O de la quantité $OO' = \frac{1}{2}OH$, en entraînant le cercle amplifié parallèlement à lui-même. On amènera de cette manière le rayon $OM_1$ à la grandeur et à la position qu'il doit prendre en $O'\mu_1$. Il reste à vérifier que les deux directions homologues $OM_1$, $O'\mu_1$ sont parallèles.

Or la droite $OM_1$ a pour coefficient d'inclinaison, par rapport à l'axe AX,

$$\frac{2R\cos(\beta-\theta_1)\sin(\alpha+\beta-\theta_1) - R\sin\alpha}{2R\cos(\beta-\theta_1)\cos(\alpha+\beta-\theta_1) - R\cos\alpha}$$
$$= \frac{2\cos(\beta-\theta_1)\sin(\alpha+\beta-\theta_1) - \sin\alpha}{2\cos(\beta-\theta_1)\cos(\alpha+\beta-\theta_1) - \cos\alpha}$$
$$= \frac{\sin(\alpha+2\beta-2\theta_1)}{\cos(\alpha+2\beta-2\theta_1)} = \operatorname{tang}(\alpha+2\beta-2\theta_1),$$

et la droite $O'\mu_1$

$$\frac{\frac{3}{2}R\cos\theta_1}{-\frac{3}{2}R\sin\theta_1} = -\cot\theta_1 = \operatorname{tang}\left(\theta_1 - \frac{\pi}{2}\right).$$

Mais l'arc $\theta_1$ est déterminé par la relation

$$\theta_1 = \frac{\alpha+2\beta}{3} - \frac{\pi}{6},$$

ce qui donne pour les coefficients angulaires des deux droites les expressions

$$\operatorname{tang}\frac{\alpha+2\beta+\pi}{3} \text{ et } \operatorname{tang}\frac{\alpha+2\beta-2\pi}{3}.$$

La différence des angles faits par les deux droites avec l'axe AX est donc égale à

$$\frac{\alpha+2\beta+\pi}{3} - \frac{\alpha+2\beta-2\pi}{3} = \pi,$$

ce qui assure le parallélisme des deux droites.

On reconnaîtra de même que, pour passer du point $M_1'$ du cercle circonscrit au point $\mu_1'$, où le rayon de courbure atteint sa valeur maximum 4R, il suffit de réduire le rayon R du premier cercle à la moitié $\frac{R}{2}$ de sa valeur et de déplacer le cercle réduit de la quantité OO' parallèlement à lui-même.

III. *Description de la courbe enveloppe considérée comme hypocycloïde.*

FIGURE 13.

Le centre O′ des circonférences C et C′ de rayon $\frac{3R}{2}$ et $\frac{R}{2}$ a pour coordonnées

$$\xi = \frac{R}{2}\cos(\alpha + 2\beta) + R\cos\alpha,$$

$$\eta = \frac{R}{2}\sin(\alpha + 2\beta).$$

Ces deux circonférences comprennent dans leur intervalle toute la courbe enveloppe. Soit R$\mu$ la pédale correspondante à une valeur particulière de l'angle $\theta$; elle fait l'angle $\theta$ avec l'ordonnée RM; soit $\mu$ le point de contact de cette pédale avec son enveloppe. La droite R$\mu$ a pour équation

$$y = -\cot\theta[x - R\cos(\alpha + 2\beta - 2\theta) - R\cos\alpha]$$

ou bien

$$y\sin\theta + x\cos\theta - R\cos(\alpha + 2\beta - 2\theta)\cos\theta - R\cos\alpha\cos\theta = 0.$$

Par le point $\mu$ menons une perpendiculaire à R$\mu$; ce sera la normale à la courbe, et déterminons les distances O′K, O′I du centre O′ à ces deux droites.

On aura d'abord

$$\begin{aligned} O'K = \delta &= \eta\sin\theta + \xi\cos\theta - R\cos(\alpha + 2\beta - 2\theta)\cos\theta - R\cos\alpha\cos\theta \\ &= \frac{R}{2}[\cos(\alpha + 2\beta)\cos\theta + \sin(\alpha + 2\beta)\sin\theta] \\ &\quad + R\cos\alpha\cos\theta - R\cos\alpha\cos\theta - R\cos(\alpha + 2\beta - 2\theta)\cos\theta \\ &= \frac{R}{2}\cos(\alpha + 2\beta - \theta) - \tfrac{1}{2}R\cos(\alpha + 2\beta - 3\theta) - \tfrac{1}{2}R\cos(\alpha + 2\beta - \theta) \\ &= -\frac{R}{2}\cos(\alpha + 2\beta - 3\theta), \end{aligned}$$

quantité à prendre en valeur absolue.

La circonférence O′ de rayon $\frac{R}{2}$ coupe au point $\lambda$ la pédale R$\mu$; on a donc O′$\lambda = \frac{R}{2}$; l'angle $\lambda$O′K est égal à $\alpha + 2\beta - 3\theta$, et par suite l'angle $\lambda$O′X′ que fait O′$\lambda$ avec O′X′, parallèle à AX, est égal à $\alpha + 2\beta - 2\theta$, c'est à dire à l'angle que fait le rayon OM avec le même axe. Les deux rayons OM et O′$\lambda$ sont donc parallèles.

La demi-corde $\lambda K$, dans le cercle $O'$ a pour valeur

$$\lambda K = \frac{R}{2}\sin(\alpha + 2\beta - 3\theta).$$

Cherchons de même la distance $O'I$ du centre $O'$ à la normale. L'équation de la normale sera, si l'on appelle $x'$ et $y'$ les coordonnées du point $\mu$,

$$(y - y')\cos\theta - (x - x')\sin\theta = 0,$$

et par conséquent la distance $\delta'$ du point $O'$ à la normale est

$$\delta' = (\eta - y')\cos\theta - (\xi - x')\sin\theta$$

expression où il faut remplacer $\eta$, $\xi$, $y'$, $x'$ par leurs valeurs en fonction de $\theta$ :

$$\eta = \tfrac{1}{2}R\sin(\alpha + 2\beta), \quad \xi = \tfrac{1}{2}R\cos(\alpha + 2\beta) + R\cos\alpha,$$

$$y' = \frac{R}{2}[\sin(\alpha + 2\beta) + 2\sin(\alpha + 2\beta - 2\theta) + \sin(\alpha + 2\beta - 4\theta)],$$

$$x' = \frac{R}{2}[\cos(\alpha + 2\beta) + 2\cos\alpha + 2\cos(\alpha + 2\beta - 2\theta) - \cos(\alpha + 2\beta - 4\theta)].$$

Il vient, toutes réductions opérées,

$$\delta' = -R\sin(\alpha + 2\beta - 3\theta) - \frac{R}{2}\sin(\alpha + 2\beta - 3\theta),$$

c'est à dire

$$\delta' = O'I = \frac{3R}{2}\sin(\alpha + 2\beta - 3\theta) = 3\lambda K.$$

Soit S le point de rencontre du rayon $O'\lambda$ prolongé, avec la normale ; on aura $\lambda S = 2O'\lambda = R$, et le point S appartient à la fois à la normale $\mu S$, à la circonférence C de rayon $\frac{3R}{2}$ et au rayon $O'\lambda S$. L'angle $S\mu\lambda$ étant droit, le point $\mu$ appartient à une circonférence de diamètre $\lambda S = R$ ; et si l'on fait pivoter cette circonférence autour du point S, le point $\mu$ décrira un élément de la courbe. La courbe enveloppe peut donc être décrite *épicycloïdalement*, ou plutôt *hypocycloïdalement*, en faisant rouler la circonférence de diamètre R au dedans de la circonférence de rayon $\frac{3R}{2}$.

La circonférence C', de rayon $\frac{R}{2}$, est l'enveloppe des positions successives de la courbe roulante.

Le rayon de courbure au point $\mu$ est égal à 8 fois la distance O'K, ou à 4 fois la longueur $S\mu$ de la normale comprise entre le point décrivant et le centre instantané de rotation.

La courbe enveloppe lieu des points $\mu$ est donc *l'hypocycloïde tricuspide* engendrée en faisant rouler le cercle de diamètre R au dedans d'un cercle de rayon $\frac{3R}{2}$, dont le centre est au point O', c'est à dire au centre du cercle des neuf points construit pour le triangle ABC.

La courbe des points $\mu$ dépend uniquement, comme forme, du rayon R du cercle circonscrit, et ne dépend pas de la forme du triangle donné ; cette forme influe seulement sur la distance OO' et sur l'orientation de cette distance, c'est à dire sur la position de l'enveloppe.

IV. *Remarque sur la construction de la courbe par points.*

FIGURE 14.

Pour construire le point $\mu$ où la pédale correspondante à un point M touche son enveloppe, abaissons du point M sur l'un des côtés AB du triangle la perpendiculaire MR, et prolongeons-la en $m'$ jusqu' à ce qu'elle recoupe la circonférence. Si l'on joint $m'$C, cette droite sera parallèle à la pédale cherchée. Car elle fait avec l'ordonnée un angle $\theta$ égal à l'angle MAC. Si donc on abaisse $M\mu'$ perpendiculaire sur $m'$C, on aura en $m'\mu'$ la longueur comprise sur la pédale entre le point R et le point de contact $\mu$. Il suffit donc pour obtenir ce point de porter parallèlement aux ordonnées la quantité $\mu'\mu = m'R$ ; c'est à dire d'ajouter géométriquement les droites $m'\mu'$, $m'R$.

La construction pratique de la courbe peut se réduire à la construction de 12 points :

1° les points correspondants au point directeur amené en coïncidence avec les 3 sommets A, B, C ;

2° les points correspondants aux trois points A', B', C', diamétralement opposés aux sommets A, B, C ;

3° les points $\mu_1$, $\mu_2$, $\mu_3$ correspondants au point directeur placé aux sommets du triangle équilatéral $M_1M_2M_3$ ; ce seront les points de rebroussement de la courbe ;

4° les points $\mu_1'$, $\mu_2'$, $\mu_3'$ correspondants au point directeur amené aux sommets du triangle équilatéral symétrique du précédent, $M_1'M_2'M_3'$; ce seront les points où la courbure de la courbe atteindra son maximum. Ces deux séries de points se trouvent respectivement sur deux circonférences concentriques.

Si l'on a déterminé avec précision l'arc de courbe compris entre deux points de rebroussement $\mu_1$, $\mu_2$, il suffira de le répéter symétriquement dans les intervalles $\mu_2\mu_3$, $\mu_3\mu_1$ pour compléter la courbe.

V. *Examens de certains cas particuliers.*

FIGURE 15.

1°. Le triangle donné est *équilatéral*, $\alpha = \beta = \dfrac{\pi}{6}$.

Les points O et O′ coïncident, les points de rebroussement $\mu_1$, $\mu_2$, $\mu_3$ correspondent aux sommets A, B, C, et les points de plus grande courbure $\mu_1'$, $\mu_2'$, $\mu_3'$ aux points diamétralement opposés sur le cercle circonscrit.

FIGURE 16.

2°. Triangle *isoscèle*. A sommet de l'angle formé par les côtés égaux ; on aura $\alpha = \beta$.

Le point de rebroussement $\mu_2$ correspond à ce sommet A ; le point $\mu_3'$ au point diamétralement opposé.

FIGURE 17.

3°. Triangle *rectangle*. A sommet de l'angle droit, $\alpha + \beta = \dfrac{\pi}{2}$.

Le point O′ est le milieu du rayon AO, puisque le sommet de l'angle droit est le point de concours des hauteurs.

Les deux cercles de rayon $\dfrac{R}{2}$ et $\dfrac{3R}{2}$, décrits du centre O′ sont tangents, l'un extérieurement, l'autre intérieurement au cercle circonscrit. La courbe enveloppe passe par les sommets B et C des angles aigus et y est tangente aux côtés BA, CA.

FIGURE 18.

4°. Triangle *évanouissant*, réduit à deux côtés AB, AC, confondus en un seul, et à la tangente à la circonférence circonscrite, formant le troisième côté infiniment petit BC.

Le problème est ramené à la recherche de l'enveloppe des droites RN, qui joignent les pieds des perpendiculaires abaissées du point M, pris sur la circonférence, sur la corde AB et la tangente BL.

Ces perpendiculaires coupent la circonférence en R′ et en N′. Si l'on porte MN″ = NN′ et MR″ = R′R sur ces perpendiculaires, la droite R″N″ sera perpendiculaire à RN, et donnera la position $\mu$ du point de contact avec l'enveloppe.

Le point de concours H des hauteurs du triangle est à la rencontre des droites BH perpendiculaire à AB, et AH perpendiculaire à la tangente BL; et le point O′ centre des cercles de rayon $\frac{R}{2}$ et $\frac{3R}{2}$ est au milieu O′ de la droite OH.

## CHAPITRE III.

### *Question incidente.*

*Relation entre les centres O et O′ des deux circonférences. Lieu géométrique.*

FIGURE 19.

Soit O′ le centre des cercles de rayon $\frac{R}{2}$ et $\frac{3R}{2}$ contenant les rebroussements et les points de courbure maximum de l'enveloppe.

Nous aurons, par rapport aux axes AX, AY,

pour O les coordonnées $x = R\cos\alpha,$

$y = R\sin\alpha,$

$\alpha$ étant l'angle OAX;

pour O′ les coordonnées $\xi = R\cos\alpha + \frac{1}{2}R\cos(\alpha + 2\beta),$

$\eta = \frac{1}{2}R\sin(\alpha + 2\beta),$

$\beta$ étant l'angle OAC, qui ajouté à OAB complète l'angle CAB = A du triangle.

On a donc $\alpha + \beta = A$, $\beta = A - \alpha$ et $\alpha + 2\beta = 2A - \alpha$;

de sorte que les coordonnées de O′ par rapport aux axes AX, AY sont

$$\xi = R\cos\alpha + \frac{R}{2}\cos(2A - \alpha),$$

$$\eta = \frac{R}{2}\sin(2A - \alpha).$$

Les coordonnées de O′ par rapport à des axes OX′, OY′ parallèles à AX, AY, menés par le point O seront données par les différences

$$\xi - x = x' = \frac{R}{2}\cos(2A - \alpha),$$

$$\eta - y = y' = \frac{R}{2}\sin(2A - \alpha) - R\sin\alpha,$$

ou bien

$$(1)\quad \begin{cases} \cos(2A - \alpha) = \dfrac{2x'}{R} = x'', \\ \sin(2A - \alpha) - 2\sin\alpha = \dfrac{2y'}{R} = y'', \end{cases}$$

en appelant $x''$ et $y''$ les coordonnées $x'$, $y'$ rapportées à la moitié du rayon R.

Imaginons que l'on fasse mouvoir l'axe AX parallèlement à lui-même, en conservant l'angle CAX = A.

L'angle $\alpha$ variera, et le côté CB aura une longueur constante ; il enveloppera dans son mouvement une circonférence concentrique au cercle O. Nous aurons l'équation du lieu décrit par le point O′ en éliminant $\alpha$ entre les deux équations (1) ; elles deviennent, en développant les sinus et cosinus,

$$(2)\quad \begin{cases} \cos 2A \cos\alpha + \sin 2A \sin\alpha = x'', \\ \sin 2A \cos\alpha - (\cos 2A + 2)\sin\alpha = y''. \end{cases}$$

Multiplions la première équation par $(\cos 2A + 2)$, la seconde par $\sin 2A$, et ajoutons, nous aurons la valeur de $\cos\alpha$

$$\cos\alpha = \frac{x''(\cos 2A + 2) + y''\sin 2A}{\cos 2A(\cos 2A + 2) + \sin^2 2A}$$

$$= \frac{x''(\cos 2A + 2) + y''\sin 2A}{1 + 2\cos 2A}.$$

On obtiendra $\sin\alpha$ en multipliant la première par $\sin 2A$, la seconde par $\cos 2A$, et en retranchant ; ce qui donne

$$\sin\alpha = \frac{x''\sin 2A - y''\cos 2A}{1 + 2\cos 2A}.$$

Elevons au carré les deux équations obtenues, et ajoutons ; $\alpha$ sera éliminé, et il vient pour équation finale

$$x''^2\sin^2 2A - 2x''y''\sin 2A\cos 2A + y''^2\cos^2 2A$$
$$+ x''^2(2 + \cos 2A)^2 + 2x''y''\sin 2A(\cos 2A + 2) + y''^2\sin^2 2A$$
$$= (1 + 2\cos 2A)^2.$$

Elle se réduit à la forme suivante

$$(5+4\cos 2A)x''^2+4\sin 2A x''y''+y''^2=(1+2\cos 2A)^2,$$

ou bien, en rétablissant $\frac{2x'}{R}$, $\frac{2y'}{R}$ à la place de $x''$ et $y''$,

$$(3)\quad y'^2+4\sin 2A x'y'+(5+4\cos 2A)x'^2=\frac{R^2}{4}(1+2\cos 2A)^2,$$

équation d'une courbe du second ordre, dont le centre est au point O.

On a d'ailleurs l'identité

$$\cos 2A = 2\cos^2 A - 1,$$

d'où l'on déduit

$$1+2\cos 2A = 4\cos^2 A - 1$$

et

$$5+4\cos 2A = 1+4(1+\cos 2A) = 1+8\cos^2 A.$$

L'équation (3) devient par conséquent

$$(4)\quad y'^2+4\sin 2A x'y'+(1+8\cos^2 A)x'^2=\frac{R^2}{4}(4\cos^2 A-1)^2.$$

Le second membre devient nul lorsque l'angle A satisfait à la relation $\cos^2 A = \frac{1}{4}$,

c'est à dire lorsque $A=\frac{\pi}{3}$ ou $\frac{2\pi}{3}$.

Le premier membre devient alors,

pour $A=\frac{\pi}{3}$,

$$y'^2+2\sqrt{3}\,x'y'+3x'^2=(y'+x'\sqrt{3})^2=0,$$

pour $A=\frac{2\pi}{3}$,

$$y'^2-2\sqrt{3}\,x'y'+3x'^2=(y'-x'\sqrt{3})^2=0.$$

Ces diverses relations définissent les valeurs du rapport $\frac{y'}{x'}$ et représentent deux droites doubles, savoir

$$y'=-x'\sqrt{3}\quad \text{pour } A=\frac{\pi}{3},$$

$$y'=+x'\sqrt{3}\quad \text{pour } A=\frac{2\pi}{3}.$$

L'ellipse lieu des points O' se réduit dans ces deux cas à une droite.

Dans tous les autres cas, le discriminant du premier membre de l'équation (4) n'est pas nul, et le calcul montre qu'il reste toujours positif et égal à $4(1-4\cos^2 A)^2$.

Si l'on fait $A = \frac{\pi}{2}$, l'équation devient $y'^2 + x'^2 = \frac{R^2}{4}$ et représente une circonférence concentrique au cercle O. C'est le cas déjà examiné du triangle rectangle BAO.

Faisons enfin $A = 0$, ce qui correspond au triangle évanouissant, formé d'une corde AB et d'une tangente ; nous aurons pour l'équation du lieu

$$y'^2 + 9x'^2 = \frac{9R^2}{4} ;$$

le lieu est une ellipse qui a pour demi-axes, parallèles aux axes coordonnés, $\frac{R}{2}$ suivant OX′ parallèle à la corde AB, et $\frac{3R}{2}$ suivant OY′.

Les axes principaux de l'ellipse générale font avec l'axe OX′ les angles

$$\phi = \frac{A}{2} \quad \text{et} \quad \phi' = \frac{A+\pi}{2} ;$$

ils sont donc parallèles aux bissectrices de l'angle A et de l'angle extérieur supplémentaire. Les longueurs des demi-axes sont, dans ces directions, données par les relations

$$r_1 = \frac{R}{2}\,\frac{4\cos^2 A - 1}{1 - 4\cos^2\frac{A}{2}}, \quad r_2 = \frac{R}{2}\,\frac{4\cos^2 A - 1}{1 - 4\sin^2\frac{A}{2}} ;$$

et la demi-distance focale $\sqrt{r_2^2 - r_1^2}$ est égale à $R\sqrt{2\cos A}$, $\cos A$ étant pris positivement.

Pour $A = \frac{\pi}{3}$ on trouve

$$\sqrt{r_2^2 - r_1^2} = R,$$

c'est le cas de l'ellipse réduite à une droite ;

et pour $A = \frac{\pi}{2}$

$$\sqrt{r_2^2 - r_1^2} = 0, \text{ cas du cercle.}$$

# CHAPITRE IV.

## I. *Rectification de la courbe enveloppe.*

FIGURE 20.

L'arc $s$ de la courbe s'obtient par l'intégration de la fonction différentielle

$$ds = \rho d\theta = -4R\cos(\alpha + 2\beta - 3\theta)d\theta$$

ce qui donne

$$s = C + \tfrac{4}{3}R\sin(\alpha + 2\beta - 3\theta),$$

avec une constante arbitraire C. Entre les limites $\theta_0$ et $\theta$, on aurait

$$s = \tfrac{4}{3}R[\sin(\alpha + 2\beta - 3\theta) - \sin(\alpha + 2\beta - 3\theta_0)].$$

Le rayon de courbure au point de la courbe qui correspond au point directeur M, s'obtient en augmentant l'arc CM de sa moitié MI portée en MI', en menant le diamètre $I'OI_1'$ et en achevant le triangle rectangle inscrit $I'I_1'I_2'$ ; on a

$$\rho = 2I_1'I_2' = 4R\cos(\alpha + 2\beta - 3\theta)$$

en valeur absolue.

Le même triangle rectangle donne

$$I_2'I' = 2R\sin(\alpha + 2\beta - 3\theta)$$

et par suite, à la constante C près, l'arc de la courbe est représenté par les $\frac{2}{3}$ de la corde $I_2'I'$. L'arc dont les extrémités correspondent à deux points M et M' est égal aux $\frac{2}{3}$ de la différence des cordes $I_2'I'$ correspondantes à ces deux points.

Appliquée à la longueur de l'arc compris entre deux rebroussements consécutifs de la courbe, la formule, où l'on fera $\theta_0 = \theta_2$, $\theta = \theta_1$, donne

$$S = \tfrac{4}{3}R[\sin(\alpha + 2\beta - 3\theta_1) - \sin(\alpha + 2\beta - 3\theta_2)] = \frac{8R}{3},$$

car le second sinus est égal à l'unité négative, et le premier à l'unité positive. La courbe est donc rectifiable, et la longueur comprise entre deux rebroussements est commensurable avec le rayon du cercle ; la courbe totale a pour longueur 8 fois le rayon R. Cette longueur est intermédiaire entre la longueur du cercle circonscrit et celle du cercle de rayon $\frac{3R}{2}$.

II. *Quadrature de la courbe.*

FIGURE 21.

Pour trouver l'aire exacte de la courbe, il faudrait intégrer la différentielle $ydx$, qui, ramenée à la variable $\theta$, s'exprimerait par une fonction entière des sinus et cosinus de cet arc. L'opération ne présenterait pas de difficulté. Mais on peut éviter ces calculs si l'on veut seulement trouver l'aire comprise entre les trois arcs $\mu_1\mu_2$, $\mu_2\mu_3$, $\mu_3\mu_1$, dont se compose la courbe totale. On y parvient approximativement de la manière suivante.

L'aire cherchée est égale à l'aire du triangle équilatéral $\mu_1\mu_2\mu_3$ de laquelle on aurait soustrait les trois segments égaux compris entre chaque arc et le côté qui lui sert de corde.

Soit $l$ la longueur du côté $\mu_1\mu_3$ ; $f$ la flèche KF de l'arc de courbe que ce côté soustend.

Si l'on rapporte ces quantités au rayon R du cercle donné, on aura

$$l = \frac{3R\sqrt{3}}{2}$$

puisque le triangle considéré est inscrit dans le cercle de rayon $\frac{3}{2}R$ ; sa surface est égale à $\frac{1}{4}l^2\sqrt{3} = \frac{27}{16}R^2\sqrt{3}$.

La flèche $f =$ KF est la différence entre IK = R et FI, qu'on peut calculer. Le point I étant le milieu de l'arc de cercle $\mu_1\mu_3$, $\mu_1$I est le côté de l'hexagone régulier inscrit, et il est par conséquent égal au rayon $\frac{3R}{2}$. Le triangle $\mu_3\mu_1$I rectangle en $\mu_1$ donne l'egalité

$$\overline{\mu_1 I}^2 = IF \times I\mu_3, \text{ c'est à dire } \frac{9R^2}{4} = IF \times 3R,$$

et par conséquent $IF = \frac{3}{4}R$ ; comme $KI = R$,

il en résulte $KF = f = \frac{R}{4}$.

FIGURE 22.

Menons au point K la tangente HH' à la courbe, et tirons les droites $\mu_3$K, $\mu_1$K.

L'aire cherchée est comprise entre le triangle $\mu_3K\mu_1$ et le trapèze HH'$\mu_1\mu_3$. Le triangle a pour mesure $\frac{1}{2}fl$; le trapèze a pour

hauteur $f$, et l'une de ses bases est égale à $l$, l'autre HH' est les $\frac{2}{3}$ de $\mu_3\mu_1$, car le point K est aux $\frac{2}{3}$ de la hauteur O'F ; le trapèze a pour mesure $\frac{1}{2}(l+\frac{2}{3}l)\times f=\frac{5}{6}fl$.

L'aire à évaluer étant comprise entre le triangle formé par des cordes, et le trapèze limité à la même base mais formé par un contour tangentiel, il est conforme à une remarque souvent verifiée d'attribuer à l'aire du segment la moyenne des deux limites, en affectant du coefficient 2 l'aire excédante, ce qui donne pour le coefficient de $lf$

$$\frac{\frac{1}{2}+2\times\frac{5}{6}}{3}=\frac{13}{18}.$$

L'aire du segment $\mu_1\mathrm{K}\mu_3$ est donc sensiblement égale à $\frac{13}{18}lf$; et comme il faut tripler ce résultat pour le retrancher de l'aire du triangle équilatéral, il vient pour l'aire cherchée

$$\mathrm{A}=\frac{27}{16}\mathrm{R}^2\sqrt{3}-3\times\frac{13}{18}\times\frac{3\mathrm{R}\sqrt{3}}{2}\times\frac{\mathrm{R}}{4}=\frac{7}{8}\mathrm{R}^2\sqrt{3}.$$

On peut comparer ce résultat aux aires des triangles équilatéraux inscrits dans les différents cercles que nous avons considérés dans notre étude, savoir les cercles de rayon R, de rayon $\frac{\mathrm{R}}{2}$, de rayon $\frac{3\mathrm{R}}{2}$.

Appelons $\mathrm{A}_1$, $\mathrm{A}_2$, $\mathrm{A}_3$ ces trois aires ; en les rangeant par ordre de grandeur, on aura

$$\mathrm{A}_1=\frac{3}{16}\mathrm{R}^2\sqrt{3},\quad \mathrm{A}_2=\frac{3}{4}\mathrm{R}^2\sqrt{3},\quad \mathrm{A}=\frac{7}{8}\mathrm{R}^2\sqrt{3},\quad \mathrm{A}_3=\frac{27}{16}\mathrm{R}^2\sqrt{3},$$

et l'on peut observer que A est le tiers de la somme $\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3$ des trois triangles rectilignes.

Si l'on assimilait la courbe $\mu_3\mathrm{K}\mu_1$ à une parabole, suivant la règle de Simpson, on aurait pour l'aire du segment

$$\frac{2}{3}\times l\times\frac{\mathrm{R}}{4}=\frac{2}{3}\times\frac{3}{2}\mathrm{R}\sqrt{3}\times\frac{\mathrm{R}}{4}=\frac{\mathrm{R}^2\sqrt{3}}{4},$$

et l'ensemble des trois segments paraboliques représenterait l'aire $\mathrm{A}_2$ du triangle équilatéral inscrit dans le cercle donné. La surface comprise au dedans de la courbe serait donc égale à la différence

$$\mathrm{A}'=\frac{27}{16}\mathrm{R}^2\sqrt{3}-\frac{3}{4}\mathrm{R}^2\sqrt{3}=\frac{15}{16}\mathrm{R}^2\sqrt{3}$$

ce qui excède de $\frac{1}{16}\mathrm{R}^2\sqrt{3}$ la valeur trouvée $\frac{7}{8}\mathrm{R}^2\sqrt{3}$.

On peut expliquer cette différence en remarquant que les tangentes à la parabole aux points $\mu_1$ et $\mu_3$ iraient rencontrer l'axe FO' de la courbe en un point L situé au milieu du segment KO'; car FL doit être égal à 2FK dans la parabole; la parabole reste donc au dessous de la courbe exacte vers les deux extrémités de l'arc $\mu_3 K \mu_1$, ce qui réduit le segment d'une certaine quantité.

Remarquons encore que

$$HH' = \frac{2}{3} l = \frac{2}{3} \times \frac{3}{2} R \sqrt{3} = R \sqrt{3}$$

est le côté du triangle équilatéral inscrit dans le cercle donné.

La longueur de l'arc $\mu_1 K \mu_3$ est comprise entre les longueurs du contour inscrit $\mu_3 K \mu_1$ et du contour tangent extérieurement $\mu_3 HH' \mu_1$; on a d'ailleurs

$$\mu_3 K = \sqrt{\frac{l^2}{4} + f^2} = \sqrt{\left(\frac{3R\sqrt{3}}{4}\right)^2 + \frac{R^2}{16}} = \frac{R}{2} \sqrt{7}.$$

Le contour inscrit $\mu_3 K \mu_1$ a donc pour développement $R \sqrt{7}$.

Le contour extérieur se compose de trois parties, dont les deux extrêmes sont égales à $\frac{R}{2}$, et la partie moyenne HH' est égale à $\frac{2}{3} l$ ou à $R \sqrt{3}$; le développement est donc égal à $R(1 + \sqrt{3})$. La longueur de l'arc est donc comprise entre ces deux limites. On a en effet les inégalités

$$R \sqrt{7} < \tfrac{8}{3} R < R(1 + \sqrt{3}),$$

c'est à dire $\quad R \times 2 \cdot 646 < R \times 2 \cdot 667 < R \times 2 \cdot 732.$

La longueur trouvée $\frac{8R}{3}$ est sensiblement égale à la moyenne entre les deux limites *en affectant du coefficient 2 la moindre valeur*; on a en effet

$$\frac{2 \cdot 646 \times 2 + 2 \cdot 732}{3} = 2 \cdot 675,$$

résultat qui excède de 0·008 seulement la valeur exacte $\frac{8R}{3}$, soit une erreur relative des $\frac{3}{1000}$ de la valeur cherchée.

## CHAPITRE V.

### *Problème inverse.*

Si l'on donne le rayon R d'une circonférence O, circonscrite à un triangle, dont les angles A, B, C sont connus, on pourra construire le triangle, en faisant autour du centre O de la circonférence des angles égaux respectivement au double 2A, 2B, 2C de chacun des angles donnés.

Nous supposerons qu'on donne en outre la position O′ du centre de deux circonférences de rayons $\frac{R}{2}$ et $\frac{3R}{2}$, et sur la plus grande des deux les points $\mu_1$, $\mu_2$, $\mu_3$, sommets d'un triangle équilatéral inscrit. Ces données suffisent pour définir entièrement *l'hypocycloïde tricuspide* E, qui aurait pour points de rebroussement les trois points $\mu_1$, $\mu_2$, $\mu_3$ et qui serait engendrée par le roulement à l'intérieur du cercle de rayon $\frac{3R}{2}$ d'un cercle de diamètre R.

Le problème à résoudre consiste à placer le cercle O et le triangle ABC de telle sorte que les pédales des divers points de la circonférence O par rapport au triangle aient pour enveloppe l'hypocycloïde E.

FIGURES 23 ET 24.

Cherchons dans le triangle ABC le point de concours H des trois hauteurs; joignons OH et soit I le milieu de la droite OH. Ce point I sera dans le cercle O la position relative du centre O′ des cercles de rayons $\frac{R}{2}$, $\frac{3R}{2}$ de sorte que le centre O du cercle de rayon R, rapporté sur le plan des cercles donnés O′, sera situé sur une circonférence $C_1$ décrite de O′ comme centre avec un rayon égal à $OI = \frac{1}{2}OH$.

Soit $O_1$ un point quelconque de cette circonférence $C_1$; de ce point comme centre avec R pour rayon on décrira la circonférence donnée, et on pourra y inscrire le triangle ABC, qui y occupera une position quelconque A′B′C′; puis on déterminera les sommets $a_1$, $b_1$, $c_1$ du triangle équilatéral inscrit qui diffère le moins du triangle A′B′C′. On sait qu'il suffit pour cela de mener par chaque sommet A′ une corde parallèle au côté opposé B′C′, et de prendre le point $a_1$ au tiers de l'arc soustendu par cette corde à partir du point A′.

Le problème serait résolu si le côté $a_1b_1$ du triangle équilatéral qu'on vient de tracer était parallèle à l'un $\mu_1\mu_2$ des côtés du triangle équilatéral inscrit dans le cercle de rayon $\frac{3R}{2}$. On satisfait à cette dernière condition en faisant tourner le cercle $O_1$ et le triangle A'B'C' qu'il renferme, autour du centre O' jusqu' à ce qu'on ait amené la droite $a_1b_1$ à être parallèle à la droite fixe $\mu_1\mu_2$. Du point O' abaissons $O'p$, $O'q$ perpendiculaires à $a_1b_1$, $\mu_1\mu_2$; joignons $O'O_1$; si nous faisons tourner la partie mobile de la figure de l'angle $pO'q$, la droite $a_1b_1$ prendra la position $a_1'b_1'$, parallèle à $\mu_1\mu_2$, et le centre $O_1$ passera au point $O_1'$; le triangle A'B'C', déplacé du même angle autour de O', sera amené dans la position où il satisfera à toutes les conditions du problème inverse.

## CHAPITRE VI.

### *Considérations cinématiques.*

Imaginons que le point directeur M parcoure la circonférence de rayon R et de centre O avec une vitesse uniforme; que dans ce mouvement il entraîne le système des trois perpendiculaires MN, ML, MR abaissées sur les côtés du triangle inscrit.

Au point M correspond sur le cercle O', de rayon $\frac{3R}{2}$, un point de contact S de la circonférence O' avec le cercle de diamètre R qui, en roulant à son intérieur, engendre comme hypocycloïde l'enveloppe de la pédale. A chaque position de M correspond une pédale RN, un point S et une position du point $\mu$, point de contact de la pédale avec son enveloppe et point décrivant appartenant à la circonférence roulante.

Les perpendiculaires MN, ML, MR constituent un système rigide qui rencontre les côtés du triangle aux points N, L, R, qu'on peut considérer comme appartenant aux côtés.

Nous nous proposons d'examiner ces divers mouvements dans ce chapitre.

Le système des droites indéfinies MN, ML, MR conserve son parallélisme, et est animé d'un mouvement de translation; chaque

point du système a une vitesse *égale* et parallèle à la vitesse du point directeur M. C'est donc un mouvement de ***translation circulaire***; chaque point décrit une circonférence de rayon R avec la vitesse $v$ du point M.

Si nous appelons $\omega$ la vitesse angulaire $\frac{d\theta}{dt}$ du rayon AM autour du sommet A pris sur la circonférence, la vitesse angulaire du rayon OM issu du centre aura pour valeur $2\omega$, et l'on aura

$$v = 2\text{R}\omega\,;$$

cette vitesse linéaire est commune à tous les points du système invariable formé par les trois perpendiculaires; l'accélération de chacun de ces points est dirigée vers le centre de la circonférence qu'il décrit, et a pour valeur $4\text{R}\omega^2$.

Les points N, L, R, pieds des perpendiculaires abaissées du point M sur les trois côtés, peuvent être considérés comme appartenant aux côtés du triangle. Le mouvement de ces points sur ces côtés est la projection du mouvement circulaire uniforme du point directeur. Ce sera donc un mouvement rectiligne ***pendulaire***, dans lequel l'accélération est proportionnelle à la distance du point mobile au milieu du côté décrit, projection du centre O sur ce même côté. Si l'on désigne par $z$ la distance de l'une des projections N, L, R au milieu du côté qu'elle parcourt, on aura pour l'équation du mouvement

$$\frac{d^2z}{dt^2} = -4\omega^2 z\,.$$

La même loi s'applique, comme nous allons le montrer, au mouvement du point $\mu$ sur sa trajectoire.

FIGURE 25.

Soit CC′ la circonférence de rayon $\frac{3\text{R}}{2}$, décrite du point O′ pris pour centre; R$\mu$ la pédale qui fait un angle $\theta$ avec l'ordonnée RR′, abaissée du point M perpendiculairement au côté AB pris pour axe des $x$.

Le point $\mu$ peut être considéré comme appartenant au cercle mobile O″ de rayon R roulant à l'intérieur de la circonférence CC′.

La pédale RN et la normale $\mu$S font des angles $\theta$ avec lex axes fixes AY et AX ; elles ont donc toutes deux la vitesse angulaire

$$\frac{d\theta}{dt} = \omega.$$

La vitesse $v$ du point $\mu$ sur sa trajectoire est égale à $\frac{ds}{dt}$, $ds$ représentant l'arc de la courbe, c'est à dire $\frac{\rho d\theta}{dt} = \rho\omega$.

On a donc, en mettant pour $\rho$ sa valeur,

$$v = 4\mathrm{R}\cos(\alpha + 2\beta - 3\theta)\omega.$$

On en déduit pour les composantes de l'accélération totale, suivant la tangente $\mu\lambda$

$$\frac{dv}{dt} = \omega\frac{d\rho}{dt} = 12\mathrm{R}\sin(\alpha + 2\beta - 3\theta)\omega^2,$$

et suivant la normale $\mu$S

$$\frac{v^2}{\rho} = \rho\omega^2 = 4\mathrm{R}\cos(\alpha + 2\beta - 3\theta)\omega^2.$$

L'accélération totale J est la résultante de ces deux composantes ; soit $\psi$ l'angle qu'elle fait avec la tangente $\mu\lambda$.

Nous aurons, en divisant $\frac{v^2}{\rho}$ par $\frac{dv}{dt}$,

$$\operatorname{tang}\psi = \tfrac{1}{3}\cot(\alpha + 2\beta - 3\theta).$$

Or l'angle $\alpha + 2\beta - 3\theta$ est donné sur la figure par l'angle $\mu$S$\lambda$. Si nous achevons le rectangle $\lambda\mu$SS' inscrit dans le cercle O'', l'angle S'$\mu\lambda$ sera le complément de $\mu$S$\lambda$, et par suite S'$\mu\lambda$ a pour tangente la cotangente de l'angle $\mu$S$\lambda$. On a donc

$$\operatorname{tang}\psi = \tfrac{1}{3}\operatorname{tang}\mathrm{S}'\mu\lambda$$

et, si l'on prend $\lambda j = \frac{1}{3}\lambda\mathrm{S}'$, la droite $\mu j$ donnera la direction de l'accélération totale J.

### Figure 26.

Considérons à part l'accélération tangentielle, et comptons les arcs $s$ sur la courbe à partir du *sommet* $\mu_1'$, c'est à dire du point où le rayon de courbure atteint sa plus grande valeur absolue 4R. Supposons que le mouvement du point $\mu$ s'opère dans le sens $\mu\mu_1'\mu_2$.

La vitesse du point $\mu$ sera égale à $\frac{ds}{dt}$ en grandeur et en signe, et l'on aura en mettant le signe en évidence

$$\frac{d^2s}{dt^2} = -12\mathrm{R}\sin(\alpha + 2\beta - 3\theta)\omega^2$$

pour l'accélération tangentielle.

Or l'arc $s$ mesuré de $\mu_1'$ à $\mu$ a pour expression

$$s = \tfrac{4}{3}\mathrm{R}\sin(\alpha + 2\beta - 3\theta)$$

puisque le facteur $\sin(\alpha + 2\beta - 3\theta)$ s'annule au point $\mu_1'$ ; on a donc

$$\frac{d^2s}{dt^2} = -9\omega^2 s = -(3\omega)^2 s$$

équation d'un *mouvement pendulaire* curviligne qui aurait pour centre attractif le point $\mu_1'$. Le coefficient de $s$ dans l'équation de ce mouvement est réglé sur une vitesse angulaire $3\omega$.

Nous avons donc constaté quatre mouvements pendulaires ; savoir

1°. ceux des trois points N, L, R mobiles sur les côtés du triangle, qui parcourent chacun dans les deux sens une longueur égale à 2R, pendant que le point directeur parcourt la circonférence entière, c'est à dire dans le temps $\mathrm{T} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$ ; il en résulte pour chacun des trois mobiles une vitesse moyenne $u$, égale au quotient

$$u = \frac{4\mathrm{R}}{\left(\frac{\pi}{\omega}\right)} = \frac{4\mathrm{R}\omega}{\pi} ;$$

2°. le mouvement du point $\mu$ sur la courbe hypocycloïdale ; dans ce mouvement le point $\mu$ parcourt l'arc $\mu_1\mu_1'\mu_2$ compris entre deux rebroussements consécutifs, pendant le temps, $\frac{\mathrm{T}}{3} = \frac{\pi}{3\omega}$, que le point directeur met à parcourir le tiers de la circonférence ; la vitesse du mobile, $u'$, est en moyenne égale au quotient

$$u' = \frac{\frac{8\mathrm{R}}{3}}{\frac{\pi}{3\omega}} = \frac{8\mathrm{R}\omega}{\pi} = 2u ;$$

elle est double de la vitesse moyenne des trois mouvements rectilignes.

La pédale a une vitesse angulaire constante, égale à $\omega$ ; les mouvements rectilignes pendulaires sont réglés sur la vitesse angulaire double, $2\omega$ ; le mouvement du point de contact sur la courbe enveloppe est réglé sur la vitesse angulaire $3\omega$. Les quatre points mobiles L, N, R, $\mu$ sont tous les quatre à un même instant sur la même droite, et l'on obtient ce théorème :

Lorsqu'on fait rouler la pédale uniformément sur l'hypocycloïde qui lui sert de courbe enveloppe, les points de rencontre de la droite mobile avec les trois côtés du triangle inscrit, parcourent ces côtés suivant la loi pendulaire $\frac{dz^2}{dt^2} = -4z\omega^2$ ; et le point de contact $\mu$ parcourt la courbe enveloppe suivant la loi $\frac{d^2s}{dt^2} = -9s\omega^2$, les $z$ étant comptés à partir des milieux des côtés décrits, et les $s$ à partir du sommet des arcs successifs de la courbe.

## The Condensation of Continuants.

By THOMAS MUIR, LL.D.

1. Rather more than twenty years ago, in a note on this subject, it was shown to the Edinburgh Mathematical Society (*Proceedings*, II., pp. 16–18) that a special form of continuant, viz., one with univarial diagonals, could be expressed by means of a similar continuant of much lower order. A new mode of proving this theorem, which has lately been hit upon, has unexpectedly led to the discovery that the peculiarity in question is not confined to this special form, but characterises continuants of any form whatever.

2. Taking first a continuant of *odd* order, viz.,

$$\begin{vmatrix} a_1 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & a_2 & b_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & a_3 & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & a_4 & b_4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & a_5 & b_5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & a_6 & b_6 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & a_7 \end{vmatrix}, \text{ or } K_{1,7} \text{ say,}$$

and multiplying it by $a_1a_3^2a_5^2a_7$ in the form

$$\begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -b_1a_3 & a_1a_3 & a_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b_3a_5 & a_3a_5 & a_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -b_5a_7 & a_5a_7 & a_5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}$$

we obtain

$$\begin{vmatrix} a_1 & . & . & . & . & . & . \\ -1 & |a_1a_2a_3| & b_2 & -b_2b_3a_5 & . & . & . \\ . & . & a_3 & . & . & . & . \\ . & -a_1 & -1 & |a_3a_4a_5| & b_4 & -b_4b_5a_7 & . \\ . & . & . & . & a_5 & . & . \\ . & . & . & -a_3 & -1 & |a_5a_6a_7| & b_6 \\ . & . & . & . & . & . & a_7 \end{vmatrix}$$

where $|a_1a_2a_3|$, $|a_3a_4a_5|$, $|a_5a_6a_7|$ are coaxial minors of $K_{1,7}$, viz.,

$$\begin{vmatrix} a_1 & b_1 & . \\ -1 & a_2 & b_2 \\ . & -1 & a_3 \end{vmatrix}, \quad \begin{vmatrix} a_3 & b_3 & . \\ -1 & a_4 & b_4 \\ . & -1 & a_5 \end{vmatrix}, \quad \begin{vmatrix} a_5 & b_5 & . \\ -1 & a_6 & b_6 \\ . & -1 & a_7 \end{vmatrix}.$$

This seven-line determinant, however, is evidently resolvable into

$$a_1a_3a_5a_7 \cdot \begin{vmatrix} |a_1a_2a_3| & -b_2b_3a_5 & . \\ -a_1 & |a_3a_4a_5| & -b_4b_5a_7 \\ . & -a_3 & |a_5a_6a_7| \end{vmatrix};$$

consequently there results

$$K_{1,7} \cdot a_3a_5 = \begin{vmatrix} |a_1a_2a_3| & b_2b_3 & . \\ a_1a_5 & |a_3a_4a_5| & b_4b_5 \\ . & a_3a_7 & |a_5a_6a_7| \end{vmatrix} \qquad \text{(I)}.$$

When the given continuant is $K_{1,2n+1}$ its co-factor on the left is $a_3a_5 \ldots a_{2n-1}$, and the continuant on the right is of the $n$th order.

3. For the case where the given continuant is of even order, no separate investigation is necessary, for, putting $a_7 = 1$, $b_6 = 0$ in the preceding result we have

$$K_{1,6} \cdot a_3a_5 = \begin{vmatrix} |a_1a_2a_3| & b_2b_3 & . \\ a_1a_5 & |a_3a_4a_5| & b_4b_5 \\ . & a_3 & |a_5a_6| \end{vmatrix} \qquad \text{(II)}$$

the want of symmetry in which at once suggests the alternative result

$$K_{1,6} \cdot a_2a_4 = \begin{vmatrix} |a_1a_2| & b_1b_2 & . \\ a_4 & |a_2a_3a_4| & b_3b_4 \\ . & a_6a_2 & |a_4a_5a_6| \end{vmatrix}. \qquad \text{(II')}$$

Putting in this last each of the $a$'s equal to $x$ and each of the $b$'s equal to $-bc$ we obtain

$$K_{1,6}\cdot x^2 = \begin{vmatrix} x^2-bc & b^2c^2 & \cdot \\ x & x(x^2-2bc) & b^2c^2 \\ \cdot & x^2 & x(x^2-2bc) \end{vmatrix}$$

and therefore in the notation of the paper of 1884

$$\begin{aligned} F(b,\, x,\, c,\, 6) &= \begin{vmatrix} x^2-bc & b^2 & \cdot \\ c^2 & x^2-2bc & b^2 \\ \cdot & c^2 & x^2-2bc \end{vmatrix} \\ &= F(b^2,\, x^2-2bc,\, c^2,\, 3) + bcF(b^2,\, x^2-2bc,\, c^2,\, 2). \end{aligned}$$

4. Since from (II′) we have

$$K_{1,8}\cdot a_2a_4a_6 = \begin{vmatrix} \lvert a_1a_2\rvert & b_1b_2 & \cdot & \cdot \\ a_4 & \lvert a_2a_3a_4\rvert & b_3b_4 & \cdot \\ \cdot & a_6a_2 & \lvert a_4a_5a_6\rvert & b_5b_6 \\ \cdot & \cdot & a_8a_4 & \lvert a_6a_7a_8\rvert \end{vmatrix}$$

it follows on putting $a_8 = 1$, $b_7 = 0$ that

$$K_{1,7}\cdot a_2a_4a_6 = \begin{vmatrix} \lvert a_1a_2\rvert & b_1b_2 & \cdot & \cdot \\ a_4 & \lvert a_2a_3a_4\rvert & b_3b_4 & \cdot \\ \cdot & a_6a_2 & \lvert a_4a_5a_6\rvert & b_5b_6 \\ \cdot & \cdot & a_4 & \lvert a_6a_7\rvert \end{vmatrix} \qquad \text{(III)}$$

which is an alternative to (I).

5. If now we take identities which give the equivalent of an even-ordered K, say $K_{1,8}$, and the equivalent of the differential quotient of this with respect to $a$, viz., the first identity of the preceding paragraph and the identity

$$K_{2,8}\cdot a_4a_6 = \begin{vmatrix} \lvert a_2a_3a_4\rvert & b_3b_4 & \cdot \\ a_6a_2 & \lvert a_4a_5a_6\rvert & b_5b_6 \\ \cdot & a_4 & \lvert a_6a_7a_8\rvert \end{vmatrix},$$

we obtain at once by division

$$a_2\cdot\left\{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots + \cfrac{b_7}{a_8}}}\right\} = \lvert a_1a_2\rvert - \frac{b_1b_2a_4}{\lvert a_2a_3a_4\rvert} - \frac{b_3b_4a_6a_2}{\lvert a_4a_5a_6\rvert} - \frac{b_5b_6a_8a_4}{\lvert a_6a_7a_8\rvert} \qquad \text{(IV)}.$$

The corresponding identity in which the number of $a$'s is odd is most readily got as before by putting in this $a_8 = 1$, $b_7 = 0$.

Substituting $x$ for each of the odd-numbered $a$'s and 1 for each of the even-numbered we find

$$x + \frac{b_1}{1 +} \frac{b_2}{x +} \frac{b_3}{1 +} \cdots + \frac{b_7}{1} = x - \frac{b_1 b_2}{x + b_2 + b_3 -} \frac{b_3 b_4}{x + b_4 + b_5 -} \frac{b_5 b_6}{x + b_6 + b_7}$$

—a result said to have been first published in 1860 by Heilermann (*Zeitschr. f. Math. u. Phys.* V. pp. 262–263).

6. Again since we also have

$$K_{1,8} \cdot a_3 a_5 a_7 = \begin{vmatrix} |a_1 a_2 a_3| & b_2 b_3 & \cdot & \cdot \\ a_1 a_5 & |a_3 a_4 a_5| & b_4 b_5 & \cdot \\ \cdot & a_3 a_7 & |a_5 a_6 a_7| & b_6 b_7 \\ \cdot & \cdot & a_5 & |a_7 a_8| \end{vmatrix}$$

and

$$K_{3,8} \cdot a_5 a_7 = \begin{vmatrix} |a_3 a_4 a_5| & b_4 b_5 & \cdot \\ a_3 a_7 & |a_5 a_6 a_7| & b_6 b_7 \\ \cdot & a_5 & |a_7 a_8| \end{vmatrix}$$

it follows by division that

$$a_3 \frac{K_{1,8}}{K_{3,8}} = |a_1 a_2 a_3| - \frac{b_2 b_3 a_1 a_5}{|a_3 a_4 a_5| -} \frac{b_4 b_5 a_3 a_7}{|a_5 a_6 a_7| -} \frac{b_6 b_7 a_5}{|a_7 a_8|}, \tag{V}$$

—that is to say, we can now express as a continued fraction not only the ratio of K to $\partial K / \partial a_1$, but also the ratio of K to $\partial^2 K / \partial a_1 \, \partial a_2$.

Further, since

$$\frac{K_{1,8}}{K_{3,8}} = \frac{K_{1,8}}{K_{2,8}} \cdot \frac{K_{2,8}}{K_{3,8}},$$

we deduce

$$a_3 \left\{ a_1 + \frac{b_1}{a_2 +} \frac{b_2}{a_3 +} \cdots + \frac{b_7}{a_8} \right\} \cdot \left\{ a_2 + \frac{b_2}{a_3 +} \cdots + \frac{b_7}{a_8} \right\}$$

$$= |a_1 a_2 a_3| - \frac{b_2 b_3 a_1 a_5}{|a_3 a_4 a_5| -} \cdots - \frac{b_6 b_7 a_5}{|a_7 a_8|}. \tag{V'}$$

7. From (IV) by putting each of the $b$'s equal to 1 we have

$$a_2\left\{a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\right\}=(a_1a_2)-\frac{a_4}{(a_2a_3a_4)-}\,\frac{a_6a_2}{(a_4a_5a_6)-}\,\frac{a_8a_4}{(a_6a_7a_8)-}\cdots$$

where $(a_1a_2)$, $(a_2a_3a_4)$, ... are now "simple" continuants. This has a special interest when the continued fraction on the left is the representative of a quadratic surd. Knowing, for example, that

$$\sqrt{13}=3+\frac{1}{\dot{1}+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\dot{6}+}\cdots$$

we conclude from it that

$$\sqrt{13}=4-\frac{1}{\dot{3}-}\,\frac{6}{13-}\,\frac{1}{13-}\,\frac{6}{3-}\,\frac{1}{\dot{8}-}\cdots.$$

The convergents to the former continued fraction are

$$3,\ 4,\ 3\tfrac{1}{2},\ 3\tfrac{2}{3},\ 3\tfrac{3}{5},\ 3\tfrac{20}{33},\ 3\tfrac{23}{38},\ 3\tfrac{43}{71},\ldots$$

and to the latter

$$4,\quad 3\tfrac{2}{3},\quad 3\tfrac{20}{33},\quad 3\tfrac{43}{71},\ldots$$

the $r$th convergent in the second case being the same as the $2r$th convergent in the first.

The process of condensation may, of course, be continued indefinitely. In the case of $\sqrt{13}$ the next result in order is

$$\sqrt{13}=\frac{11}{3-}\,\frac{26}{\dot{426}-}\,\frac{144}{251-}\,\frac{1014}{226-}\,\frac{144}{476-}\,\frac{234}{\dot{51}-}\cdots$$

where the 1st, 2nd, 3rd, .. convergents are the 4th, 8th, 12th, ... of the original continued fraction.

*Second Meeting, 9th December* 1904.

Mr W. L. THOMSON, President, in the Chair.

## The Proof by Projection of the Addition Theorem in Trigonometry.

By D. K. PICKEN, M.A.

The object of this paper is to remove the difficulty that arises in giving a general proof by projection methods of this theorem, without in any way interfering with the single-valuedness of the position of a radius vector tracing out angles from a given initial position, when the values of the trigonometrical ratios are given.

It is necessary, first of all, to give a clear statement of the definitions and theorems in Projection.

*Definition*: The projection of a point S on a straight line XY is the foot Z of the perpendicular from S on XY.

As the point S moves *in any manner*, the point Z moves backwards and forwards along XY. If we call the amount of a motion of Z from X towards Y a *positive segment*, and that of a motion in the opposite direction a *negative segment*, the total displacement of Z corresponding to a given motion of S is a positive or a negative segment, which is the algebraic sum of the alternately positive and negative segments which Z describes during the motion. Also, if S is given a succession of motions, the total displacement of Z is the algebraic sum of the displacements due to the several motions (*vide* Theorem IV. *infra.*).

*Definition*: If S moves along a straight line PQ from P to Q, the positive or negative segment MN described by Z is called the projection of PQ on XY.

The following theorems are then obvious:—

I. The projection of QP on XY = – (the projection of PQ on XY).

II. If UV is equal to, parallel to and in the same direction as PQ the projection of UV on XY = the projection of PQ on XY.

III. If R be *any point of the unlimited line through* P and Q, so that PR $= n$ . PQ where $n$ is *any real number*, the projection of PR on XY $= n$ (the projection of PQ on XY); for the projection K of R lies between M and N, on MN produced, or on NM produced, according as R lies between P and Q, on PQ produced, or on QP produced.

IV. If P and Q be joined by a succession of straight lines

$$PQ_1,\ Q_1Q_2,\ \ldots\ Q_{r-1}Q,$$

the projection of PQ on XY

$=$ the sum of the projections of $PQ_1$, $Q_1Q_2$, ... $Q_{r-1}Q$ on XY.

The generality of the proof given below of the Addition Theorem depends on Theorem III.

*The Trigonometrical Ratios.*

Let angles $\Theta$ be described by the turning in one plane of a straight line OP about a fixed point O in it, from a fixed initial position OA. The words *positive* and *negative* can then obviously be applied to distinguish the two kinds of turning. Let OB be the position of OP when $\Theta$ is *a positive right angle*, and let AO, BO produced meet the circle described by P in A′ and B′.

*Definitions*:

The ratio (projection of OP on B′OB : length of OP) is called the *sine* of $\Theta$.

The ratio (projection of OP on A′OA : length of OP) is called the *cosine* of $\Theta$; etc., etc.

It follows that these trigonometrical ratios are single-valued functions of the position of the vector OP, and that when $\sin\Theta$ and $\cos\Theta$ are given the position of OP is uniquely defined.

If OQ, OQ′, OQ″ are the positions of OP when $\Theta = a$, $-a$ and $\left(a + \frac{\pi}{2}\right)$, it is easy to obtain from consideration of the relative positions of Q, Q′, Q″ on the circle, general proofs of the formulae:

$$\sin(-a) = -\sin a\,;\ \cos(-a) = \cos a\,;$$

$$\sin\left(a + \frac{\pi}{2}\right) = \cos a\,;\ \cos\left(a + \frac{\pi}{2}\right) = -\sin a.$$

*The Addition Theorem.* (Fig. 27.)

Let $OA_1$ be the position of OP when $\Theta = \alpha$, $OB_1$ when $\Theta = \alpha + \frac{\pi}{2}$; and let angles $\phi$ be measured by the turning of OP from the initial position $OA_1$. Let OQ be the position of OP when $\phi = \beta$; then OQ is the position of OP when $\Theta = \alpha + \beta$. Let $M_1$, $N_1$ be the projections of Q on $A_1'OA_1$ and $B_1'OB_1$.

We have then

$OQ\cos(\alpha + \beta)$

$= $ projection of OQ on A′OA

$= $ (projection of $OM_1$ + projection of $M_1Q$) on A′OA [Thm. IV.]

$= $ (projection of $OM_1$ + projection of $ON_1$) on A′OA [Thm. II.]

$= \{$projection of $(\cos\beta . OA_1)$ + projection of $(\sin\beta . OB_1)\}$ on A′OA

$= \{\cos\beta$(projection of $OA_1$) + $\sin\beta$(projection of $OB_1$)$\}$ on A′OA [Thm. III.]

$$= \cos\beta . (OA_1\cos\alpha) + \sin\beta\left\{OB_1\cos\left(\alpha + \frac{\pi}{2}\right)\right\};$$

$$\therefore \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta + \cos\left(\alpha + \frac{\pi}{2}\right)\sin\beta$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Similarly, by projecting on B′OB, we get

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\left(\alpha + \frac{\pi}{2}\right)\sin\beta$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and the theorems are true whatever be the sign and whatever the magnitude of the angles $\alpha$ and $\beta$.

# The Turning-Values of Cubic and Quartic Functions and the Nature of the Roots of Cubic and Quartic Equations.

By P. PINKERTON, M.A.

## THE CUBIC.

Let a cubic function be denoted by

$$z^3 + a_1 z^2 + b_1 z + c_1.$$

Put $z = x - \dfrac{a_1}{3}$ and the function takes the form

$$x^3 + 3qx + r.$$

Consider the graph of this function, represented by

$$y = x^3 + 3qx + r. \quad - \quad - \quad - \quad (1)$$

If we shift the origin to the point $(h, k)$ the equation of the graph becomes

$$\begin{aligned} \eta + k &= (\xi + h)^3 + 3q(\xi + h) + r \\ &= \xi^3 + 3\xi^2 h + 3\xi(h^2 + q) + h^3 + 3qh + r. \end{aligned}$$

The point $(h, k)$ is a turning-point if

$$h^2 + q = 0 \quad - \quad - \quad - \quad - \quad - \quad (2)$$

$$\begin{aligned} \text{and} \qquad k &= h^3 + 3qh + r \\ &= h(h^2 + q) + 2qh + r \\ &= 2qh + r \,; \quad - \quad - \quad - \quad - \quad (3) \end{aligned}$$

for the equation of the graph is then

$$\eta = \xi^3 + 3h\xi^2, \quad - \quad - \quad - \quad - \quad (4)$$

and, near the origin, the graph has approximately the same shape as the parabola $\qquad \eta = 3h\xi^2,$

so that, at the origin, there is a minimum turning-point if $h$ is positive, and a maximum turning-point if $h$ is negative.

From (2) and (3) $\quad (k - r)^2 = 4q^2h^2 = -4q^3\,;$

$\therefore$ the turning-points are given by

$$h^2 + q = 0 \quad - \quad - \quad (2)$$

$$\text{and} \quad k^2 - 2rk + (r^2 + 4q^3) = 0. \quad - \quad - \quad (5)$$

These are quadratic equations; therefore there are two, and only two turning-points on the graph of a cubic function:

(*a*) the two turning-points coincide if $q=0$, when equation (4) becomes $\eta=\xi^3$, whose form is well known (Fig. 28 (*a*));

(*b*) the two turning-points are imaginary if $q$ is positive, in which case it is easy to see that $y=3qx+r$ is an inflexional tangent (Fig. 28 (*b*));

(*c*) there are two real turning-points if $q$ is negative (Fig. 28 (*c*)).

Hence the $x$-axis may cut the graph in three coincident points in case (*a*); or in one real point and two imaginary points in cases (*a*), (*b*), (*c*); or in three real points in case (*c*), when two of the points may coincide.

Correspondingly, the roots of the equation

$$x^3+3qx+r=0$$

will be all equal if $q=0$ and $r=0$;
one of the roots will be real and two imaginary,

if $q=0$,
if $q$ is positive, in case (*b*),
if the product of the roots of equation (5) is positive, in case (*c*),
*i.e.*, if $r^2+4q^3$ is positive.

Hence $r^2+4q^3$ positive is a general criterion for one real and two imaginary roots.

The roots will be all real and different if the roots of equation (5) are unlike in sign, *i.e.*, if $r^2+4q^3$ is negative.

There will be two equal roots if a value of $k$ as determined from equation (5) is zero, *i.e.*, if $r^2+4q^3$ is zero.

The discussion of the cubic function

$$Az^3+a_1z^2+b_1z+c_1$$

and of the corresponding equation is clearly contained in the above.

### The Quartic.

Let a quartic function be denoted by

$$z^4+a_1z^3+b_1z^2+c_1z+d_1.$$

Put $z=x-\dfrac{a_1}{4}$ and the function takes the form

$$x^4+6qx^2+4rx+s.$$

Consider the graph of the function represented by

$$y = x^4 + 6qx^2 + 4rx + s.$$

Shift the origin to the point $(h, k)$ and the equation of the graph becomes

$$\begin{aligned}\eta + k &= (\xi + h)^4 + 6q(\xi + h)^2 + 4r(\xi + h) + s \\ &= \xi^4 + 4h\xi^3 + 6\xi^2(h^2 + q) + 4\xi(h^3 + 3qh + r) + (h^4 + 6qh^2 + 4rh + s).\end{aligned}$$

Now choose $h^3 + 3qh + r = 0$ - - - - - - (6)

$$\begin{aligned}\text{and}\quad k &= h^4 + 6qh^2 + 4rh + s \\ &= h(h^3 + 3qh + r) + 3qh^2 + 3rh + s \\ &= 3qh^2 + 3rh + s. \qquad - - - - - (7)\end{aligned}$$

The equation of the graph then becomes

$$\eta = \xi^4 + 4h\xi^3 + 6\xi^2(h^2 + q). \qquad - - (8)$$

The equation (6) has for its roots the abscissae of the turning-points of the graph, and (7) gives the corresponding values of the ordinates. The values of the ordinates are expressed as the roots of an equation in $k$ by eliminating $h$ between (6) and (7).

Multiplying (6) by $3q$ and (7) by $h$, and subtracting, we get

$$3rh^2 + (s - k - 9q^2)h - 3qr = 0. \qquad - - (9)$$

Solving equations (7) and (9) for $h^2$ and $h$ we get

$$h^2 = \frac{9qr^2 + (s - k)(s - k - 9q^2)}{9r^2 - 3q(s - k - 9q^2)},$$

$$h = \frac{3r(s - k) + 9q^2r}{3q(s - k - 9q^2) - 9r^2},$$

whence, after reduction,

$$\begin{aligned}&k^3 - 3k^2(s - 6q^2) + 3k(s^2 - 12sq^2 + 27q^4 + 18qr^2) \\ &\qquad - (s^3 - 18q^2s^2 + 81sq^4 - 54q^3r^2 - 27r^4 + 54qr^2s) = 0. \qquad (10)\end{aligned}$$

It is convenient to denote the absolute term by $\Delta$.

(*a*) The three roots of the cubic equation

$$h^3 + 3qh + r = 0 \qquad - - \quad (6)$$

may be all equal. Since the coefficient of $h^2$ in the equation is zero, each root is zero; $\therefore$ $q = 0$ and $r = 0$ and the equation of the graph becomes, by (8), $\qquad \eta = \xi^4.$

The graph is now easily traced. (Fig. 29 ($\alpha$)).

($\beta$) Two of the roots of the cubic equation

$$h^3 + 3qh + r = 0$$

may be equal. We know that in that case

$$r^2 + 4q^3 = 0;$$

$$\therefore \quad 4q^3h^3 - 3qr^2h - r^3 = 0;$$

$$\therefore \quad (4q^2h^2 + 4qrh + r^2)(qh - r) = 0;$$

$$\therefore \quad h = -\frac{r}{2q},\ -\frac{r}{2q},\ \text{or}\ \frac{r}{q}.$$

Taking $h = -\dfrac{r}{2q}$ and $k = 3qh^2 + 3rh + s = 3q^2 + s$,

equation (8) becomes $$\eta = \xi^4 - \frac{2r}{q}\cdot\xi^3.$$

Hence the $\xi$-axis is an inflexional tangent at the origin, and there is a turning-point between $\xi = 0$ and $\xi = \dfrac{2r}{q}$. To determine the turning-point we notice that $\eta$ has a turning-value when

$$\xi\cdot\xi\cdot\xi\left(-3\xi + \frac{6r}{q}\right)$$

has a turning-value, which occurs when $\xi = -3\xi + \dfrac{6r}{q}$, since the sum of the factors is constant, *i.e.*, when $\xi = \dfrac{3r}{2q}$; and therefore when $\eta = -\dfrac{27}{16}\cdot\dfrac{r^4}{q^4}$. (Fig. 29 ($\beta$)).

($\gamma$) There may be two imaginary roots of the equation

$$h^3 + 3qh + r = 0. \quad - \quad - \quad - \quad (6)$$

Take $h$ equal to the real root of this equation

and $$k = 3qh^2 + 3rh + s.$$

Then the equation of the graph by (8) is

$$\eta = \xi^2\{(\xi + 2h)^2 + 2(h^2 + 3q)\}.$$

Now $h^2 + 3q = -\dfrac{r}{h}$.

But the product of the three roots of equation (6) is $-r$ and the product of the two imaginary roots is positive;

$\therefore \quad h$ and $r$ have opposite signs;

$\therefore \quad h^2 + 3q$ is positive, $= \dfrac{m^2}{2}$, say,

and the equation of the graph can be written in the form

$$\eta = \xi^2\{(\xi + 2h)^2 + m^2\},$$

from which the form of the graph is easily seen (Fig. 29 ($\gamma$)).

($\delta$) The roots of the equation

$$h^3 + 3qh + r = 0$$

may be all real.

The sum of these roots is zero; therefore there is a positive root and a negative root. Taking $h$ to be a root of the same sign as $r$, if $r \neq 0$, so that $h^2 + 3q = -\dfrac{r}{h} = -\dfrac{n^2}{2}$, say, the equation of the graph takes the form

$$\eta = \xi^2\{(\xi + 2h)^2 - n^2\},$$

from which the form of the graph is easily seen (Fig. 29 ($\delta$)).

If $r = 0$, take $h = 0$ and the equation becomes $\eta = \xi^2\{\xi^2 + 6q\}$; where, of course, $q$ is negative since $r^2 + 4q^3$ is negative. (Fig. 29 ($\delta'$)).

Now consider the equation

$$x^4 + 6qx^2 + 4rx + s = 0.$$

1. This equation will have four equal roots if

$$\eta = -k \text{ and } \eta = \xi^4$$

intersect in four coincident points.

$$\therefore \quad q = 0 \text{ and } r = 0.$$

But $k = 0$ also, $\therefore \ s = 0$.

Hence the criterion is $\quad q = 0, \ r = 0, \ s = 0.$

2. The equation will have three equal roots if

$$\eta = -k \text{ and } \eta = \xi^4 - \frac{2r}{q} \cdot \xi^3$$

intersect in three coincident points, and a fourth different point.

$$\therefore \quad r^2 + 4q^3 = 0, \text{ and } k = 0.$$

But it has been shown that $k = s + 3q^2$;

$\therefore$ the conditions are that

$$r^2 + 4q^3 = 0 \text{ and } s + 3q^2 = 0.$$

3. The equation has two pairs of roots equal when the graph is of the form shown in Fig. 29 ($\delta'$) and when $\eta = -k$ is a tangent at two of the turning-points of
$$\eta = \xi^2\{\xi^2 + 6q\}.$$

Hence $r = 0$ and two of the roots of equation (10) vanish;

$$\therefore \quad s^2 - 12sq^2 + 27q^4 = 0$$

$$\text{and} \quad s^3 - 18q^2s^2 + 81sq^4 = 0;$$

$\therefore$ $s - 9q^2 = 0$ is both necessary and sufficient.

$\therefore$ the conditions are $$r = 0 \text{ and } s = 9q^2.$$

4. The equation has two roots equal if one root of equation (10) is zero;
$$\therefore \quad \Delta = 0.$$

5. The equation has two real and two imaginary roots if the product of the roots of equation (10) is negative;
$$\therefore \quad \Delta \text{ is negative.}$$

6. The equation has its roots all real and different if the product of the roots of equation (10) is positive;
$$\therefore \quad \Delta \text{ is positive.}$$

7. The equation has its roots all imaginary, if the product of the roots of equation (10) is positive;
$$\therefore \quad \Delta \text{ is positive.}$$

It remains to distinguish the last two cases.

In case 6, $r^2 + 4q^3$ is negative as only form Fig. 29 ($\delta$) or ($\delta'$) of the graph is possible. In case 7, $r^2 + 4q^3$ may be either negative or positive, or zero, since the graph may have any of the possible forms. If, then, $q$ is positive and $\Delta$ is positive all the roots are imaginary; but, if $q$ is negative, we observe that the roots of equation (10) are all positive if the roots under discussion are all imaginary, and the roots of equation (10) are two negative and one positive if the roots under discussion are all real; therefore, only if these roots are all imaginary, can the order of the signs of the terms of equation (10) be $+ - + -$. Hence if $q$ is positive or zero and $\Delta$ positive, the roots are all imaginary; if $q$ is negative, $\Delta$ positive, $s - 6q^2$ positive, and $s^2 - 12sq^2 + 27q^4 + 18qr^2$ positive, the roots are all imaginary; otherwise, if $\Delta$ is positive, the roots are all real.

## Démonstrations de deux théorèmes de Géométrie.

Par M. Edouard Collignon,
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### Figure 30.

1. Soit ABCD un rectangle, AC l'une de ses diagonales; si l'on prend un point M sur la diagonale et qu'on mène les parallèles aux côtés, on a deux rectangles AEFD, AGHB équivalents.

Soit en effet AB une force, AD une autre force, AC sera la résultante; et si l'on prend les moments des deux forces par rapport à un point M de la résultante on aura

$$AB \times ME = AD \times MG,$$

ce qui démontre le théorème.

### Figure 31.

2. Soit ADB un triangle et C un point sur le côté AB; on aura

$$DA^2 . BC + DB^2 . AC - DC^2 . AB = AB . AC . CB.$$

Plaçons en A une masse $m$ proportionnelle au segment CB, et en B une masse $m'$ proportionnelle au segment AC.

Le centre de gravité des deux masses $m$ et $m'$ sera le point C. En effet, on aura

$$\frac{m}{m'} = \frac{CB}{CA}.$$

Le moment d'inertie $I_0$ des masses $m$, $m'$ par rapport au point C est égal à $(m+m') \times AC . CB$, et si l'on passe du point C au point D, le moment d'inertie I par rapport à D sera

$$I = I_0 + (m+m')DC^2,$$

c'est-à-dire,

$$m . DA^2 + m' . DB^2 = (m+m')AC . CB + (m+m')DC^2,$$

ce qui revient à la suivante

$$DA^2 . CB + DB^2 . AC = AB . AC . CB + AB . DC^2.$$

*Third Meeting, 13th January* 1905.

Mr W. L. THOMSON, President, in the Chair.

## Contact between a Curve and its Envelope.

By EDWARD B. ROSS, M.A.

This paper deals with a few of the simpler specialisations of the intersections of a plane curve and the envelope of the family to which it belongs. It follows the method adopted by Professor Chrystal in dealing with the $p$-discriminant of a differential equation of the first order. This method is specially applicable to definite problems; in these it is safer to work out the result than to rely on theory.

A summary of the results is given at the end.

The family is taken to be $\phi(x, y, t) = 0$; $t$ is the parameter. The $t$-discriminant locus $\Delta_t = 0$, with which rather than with the somewhat vague envelope we must deal, is given by

$$\phi = 0, \quad \phi_t = 0.$$

$\Delta_t = 0$ should be found by a formal process, not by a short cut.

Our work will be simplified if we take the curve under consideration to be $t = 0$; the point where it meets the envelope to be $x = 0$, $y = 0$; and the tangent to it there the $x$-axis. We assume that $\phi$ is expansible in powers of $t$, $x$, $y$ in the neighbourhood of the point $x = 0$, $y = 0$, $t = 0$.

Let $$\phi = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \dots,$$

where
$$A_0 = a_0 + b_0 x + c_0 y + d_0 x^2 + e_0 xy + f_0 y^2 + \dots,$$
$$A_1 = a_1 + b_1 x + c_1 y + \dots,$$
$$A_2 = a_2 + b_2 x + \dots,$$
$$A_3 = a_3 + \dots,$$
etc., etc.

Since $\phi(0)=0$, *i.e.*, $A_0=0$ touches the $x$-axis at the origin, $a_0=0$, $b_0=0$; as $\Delta_t=0$ also passes through the origin, $a_1=0$.

A sufficient approximation to the envelope in the ordinary case is

$$4a_2(c_0y+d_0x^2)=(b_1x)^2.$$

This shows ordinary 2-pointic contact between the envelope and the curve. This approximation is not sufficient if $b_1$, $a_2$, or $c_0$ be zero.

We shall find it useful to write $[\lambda]$ for a quantity of order $\lambda$ in the variable $x$ of the *envelope*, in terms of which we may suppose $y$ and $t$ expanded. If $A_0$ is of order 4 in this variable (and $c_0 \neq 0$), the expansions of $y$ in terms of $x$ derived from $A_0=0$ and from the envelope would first differ in the coefficient of $x^4$, and so the curves would have 4-pointic contact; if $c_0=0$, more information is required before the species of contact can be assigned.

Take $\alpha_0$, $\alpha_1$, ... as the orders of $A_0$, $A_1$, ..., $\eta$ of $y$, $\tau$ of $t$.

$$\phi=0 \text{ or } 0=A_0+A_1t+A_2t^2+\dots,$$

may be written $\quad 0=[\alpha_0]+[\alpha_1+\tau]+[2\tau]+\dots,$

and $\phi_t=0$, $\quad 0=\qquad [\alpha_1]+[\tau]+\dots\qquad,$

provided $a_2 \neq 0$.

(1) Let $b_1=0$. We get $\alpha_1=\tau$, $\alpha_0=2\tau$. In the simplest case $\tau=2$ so that $\alpha_0=4$.

So we do not get in this way 3-pointic contact. By the theory of implicit functions if $a_2$, $c_0 \neq 0$, $y$ is expansible in integral powers of $x$; so $\eta$ and $\alpha_1$ are integral, and $\alpha_0$ even. So from this case we can get only even-pointic contact.

(2) Let $a_2=0$, $b_1 \neq 0$.

Our equations take the form

$$0=[\alpha_0]+[\alpha_1+\tau]+[2\tau+1]+[3\tau]+\dots,$$
$$0=\qquad [\alpha_1]\ +\ [\tau+1]+[2\tau]+\dots,$$

$\tau=\frac{1}{2}$, $\alpha_1=1$, $\alpha_0=\frac{3}{2}$.

There is a cusp on the envelope, the approximation being

$$4(b_1x)^3+27a_3(c_0y)^2=0;$$

so that it breaks down if $b_1$, $c_0$, or $a_3=0$.

(3) Let $a_2 = 0$, $b_1 = 0$.

We find $a_0 = 3$; the algebraic approximation is

$$-4A_1^3A_3 + A_1^2A_2^2 + 18A_0A_1A_2A_3 - 4A_0A_2^3 - 27A_0^2A_3^2 = 0.$$

From this we find that there are two branches of the envelope each having 3-pointic contact with the curve. In some cases the branches coincide, *e.g.*, a curve enveloping its circles of curvature.

The condition for 4-pointic contact between the branches appears to be

$$b_2^2c_0 - 3a_3(d_1c_0 - c_1d_0) = 0.$$

In the important particular case of a family of straight lines, one of the branches is accurately $y = 0$, and the other is approximately $(4b_2x)^3 + 27a_3^2c_0y = 0$, which appears to reduce to $y = 0$ accurately for $b_2 = 0$.

(4) Let $c_0 = 0$. This corresponds to a double point on the original curve; the envelope has the double-point indicated by

$$4a_2A_0 = A_1^2.$$

If $d_0 = 0$ and $b_1 = 0$, one tangent of the envelope coincides with one of the curve.

(5) If the double point be a cusp,

$$(c_0 = 0 \text{ and say } d_0 = 0,\ e_0 = 0),$$

the cuspidal tangent bisects the angle between the tangents to the envelope

$$4a_2(f_0y^2 + g_0x^3) = (b_1x)^2.$$

When in addition $b_1 = 0$,

$$4a_2(f_0y^2 + g_0x^3) = (c_1y)^2$$

indicates that the envelope has a cusp with the same cuspidal tangent.

(6) If in (4) $c_1$ also is zero,

$$4a_2A_0 = A_1^2;$$

*i.e.*, the two branches of the discriminant have 3-pointic contact with the branches of the curve.

In (5) if $c_1 = 0$ the envelope coincides more closely with the curve.

(7) If $c_0 = 0$, $a_2 = 0$,
we get for the envelope, supposing $a_3 \neq 0$,

$$4\mathrm{A}_1^3 + 27 a_3 \mathrm{A}_0^2 = 0,$$

*i.e.*, $\mathrm{A}_1$ is of order $\frac{4}{3}$.

(8) If $c_0 = 0$, $d_0 = 0$, $b_1 = 0$, $a_2 = 0$,
the envelope has three branches touching the $x$-axis.

## Results.

We may now write down our results in a form independent of our choice of origin and axes.

(1) If in addition to $\phi = 0$, $\phi_t = 0$,

$$\begin{vmatrix} \phi_x, & \phi_y \\ \phi_{xt}, & \phi_{yt} \end{vmatrix} = 0, \quad - \quad - \quad \text{(i)}$$

we have 4-pointic contact.

(2) If $\phi_{tt} = 0$ (ii), the envelope has a cusp.

(3) If conditions (i) and (ii) hold, the envelope has two branches which have each 3-pointic contact with the curve.

(4) To a double point $\phi_x = 0$, $\phi_y = 0$ $\quad - \quad - \quad - \quad$ (iii)
corresponds a double-point.

If, in addition to (iii),

$$\begin{vmatrix} \phi_{xx}, & 2\phi_{xy}, & \phi_{yy} \\ \phi_{tx}, & \phi_{ty}, & \\ & \phi_{tx}, & \phi_{ty} \end{vmatrix} = 0, \quad - \quad - \quad \text{(iv)}$$

one tangent is common.

(5) The tangent at a cusp,

$$\text{(iii) with } \begin{vmatrix} \phi_{xx}, & \phi_{xy} \\ \phi_{xy}, & \phi_{yy} \end{vmatrix} = 0 \quad - \quad \text{(v)},$$

bisects the angle between the tangents at the double point on the envelope.

If in addition (iv) holds,

*i.e.*, (iii) and
$$\left\| \begin{array}{ll} \phi_{xx}, & \phi_{xy} \\ \phi_{xy}, & \phi_{yy} \\ \phi_{tx}, & \phi_{ty} \end{array} \right\| = 0 \quad - \quad - \quad \text{(vi)},$$
the envelope has a cusp with the same cuspidal tangent.

(6) If in addition to (iii)
$$\phi_{tx} = 0, \ \phi_{ty} = 0, \quad - \quad - \quad - \quad \text{(vii)}$$
(*i.e.*, (i) and (iii)) the branches of the discriminant have 3-pointic contact with those of the curve.

(7) If (ii) and (iii) hold, the envelope has a singularity of the form $\eta^3 = \lambda\xi^4$, where $\eta = 0$ is the tangent to $\phi_t = 0$.

(8) But if this tangent should coincide with one of the two tangents to the curve at the double-point, *i.e.*, (iv), the form is $\eta = \lambda\xi^2$ thrice.

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## A Proof of the Theorem that the Arithmetic Mean of *n* positive quantities is not less than their Harmonic Mean.

By W. A. Lindsay, M.A., B.Sc.

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## Two Theorems on the factors of $2^p - 1$.

By George D. Valentine, M.A.

*Fourth Meeting, 10th February* 1905.

Mr W. L. THOMSON, President, in the Chair.

## The Ambiguous Cases in the Solution of Spherical Triangles.

By T. J. I'A BROMWICH.

Although the following note makes no pretence at novelty so far as the results are concerned, yet the method employed does not seem to occur in the ordinary text-books on Spherical Trigonometry. I have found this process very useful in explaining to beginners how to distinguish between the various possibilities, and I hope it may be of some interest to other teachers.

As a starting point, suppose the known parts (two sides and an opposite angle) to be $a$, $b$, A; we have then the equation for $c$

$$\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A}.$$

When $c$ is known, the triangle is determined without further ambiguity, since the angles are uniquely determined when the sides are known; thus the only ambiguity arises through the determination of $c$ from the given parts.

Write now $t = \tan\frac{1}{2}c$, and after a little reduction we find the quadratic for $t$:

$$(1) \qquad (\cos a + \cos b)t^2 - 2t \sin b \cos \mathrm{A} + (\cos a - \cos b) = 0.$$

If we suppose, as is usual in the elementary theory, that all sides and angles are less than two right angles, it follows that the admissible roots of (1) are limited by the further condition of being *real* and *positive*.

Take first the case when $(\cos a + \cos b)$ and $(\cos a - \cos b)$ have *opposite* signs; then the two roots of (1) are real, but only one of them is positive.

But $\qquad (\cos a + \cos b)(\cos a - \cos b) = (\sin b - \sin a)(\sin b + \sin a),$

so that, since $\sin a$ and $\sin b$ are positive, this case occurs if (and only if) $\sin a$ is greater than $\sin b$. Hence :—

*If* $\sin a > \sin b$, *one and only one triangle exists with the given parts.*

If $\sin a$ is less than $\sin b$, the quadratic (1) has real roots only if

$$\sin^2 b \cos^2 A \geqq \cos^2 a - \cos^2 b$$

or, if $$\sin^2 a \geqq \sin^2 b \sin^2 A.$$

Under the restriction already assumed as to sides and angles, the last condition is equivalent to

$$\sin a \geqq \sin b \sin A.$$

The roots $t_1$, $t_2$ of the quadratic have then the same sign, which is the sign of

$$\tfrac{1}{2}(t_1 + t_2) = \sin b \cos A/(\cos a + \cos b).$$

Now $(\cos a + \cos b)$ and $(\cos a - \cos b)$ have the same sign, which must therefore be the same as the sign of $\cos a$; and consequently $t_1$, $t_2$ have the same sign as $\cos A/\cos a$. Thus we have the result :—

*If* $\sin b > \sin a \geqq \sin b \sin A$, *two triangles exist with the given parts, provided that* $\cos A/\cos a$ *is positive; the two triangles being coincident in case* $\sin a = \sin b \sin A$. *But if* $\sin b > \sin a$, *and either of the other conditions is broken, there is no triangle with the given parts.*

We have now exhausted all cases except those for which

$$\sin a = \sin b.$$

Then $$(\cos a + \cos b)(\cos a - \cos b) = 0.$$

If $\cos a = \cos b$, the roots of the quadratic (1) are 0 and $\sin b \cos A/\cos a$; thus there is one triangle if $\cos A/\cos a$ is positive, and no triangle otherwise.

We find the same result if $\cos a = -\cos b$. Thus :—

*If* $\sin a = \sin b$, *there is one triangle with the given parts, if* $\cos A/\cos a > 0$; *no triangle if* $\cos A/\cos a \leqq 0$.

The tests obtained for the cases $\sin a \lesseqgtr \sin b$, imply that $\cos a$ is not zero; if it happens that $a$ is a right angle, $b$ must also be a right angle so as to satisfy $\sin a \leqq \sin b$. In this case the quadratic (1) reduces to

$$t \cos A = 0$$

implying that there is no such triangle unless A is also a right angle. Thus :—

*The only case not included in the previous tests is* $a = b = \tfrac{1}{2}\pi$; *and there can then be no triangle with the given parts, unless* $A = \tfrac{1}{2}\pi$; *if* $A = \tfrac{1}{2}\pi$, *the triangle is indeterminate.*

We have now completed the discussion of the first ambiguous case. To deal with the second ambiguous case (two angles and an opposite side, say, A, B, $a$, being given), we may start from the relation

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

and transform it to the quadratic

$$(2) \qquad (\cos A - \cos B)T^2 - 2T \sin B \cos a + (\cos A + \cos B) = 0$$

by taking $T = \tan\frac{1}{2}C$. But the results can be obtained more quickly by using those already found, and transforming them by the polar triangle properties.

In tabular form, the results are* :—

$a$, $b$, A given

| | |
|---|---|
| Two Triangles - - | $\sin b > \sin a > \sin b \sin A$ *and* $\cos A/\cos a > 0$ |
| Two coincident Triangles | $\sin a = \sin b \sin A$ *and* $\cos A/\cos a > 0$ |
| One Triangle - - | (i) $\sin a > \sin b$<br>(ii) $\sin a = \sin b$ *and* $\cos A/\cos a > 0$ |
| Infinity of Triangles - | $a = b = A = \frac{1}{2}\pi$ |
| No Triangle - - - | Any case not included in the preceding |

If A, B, $a$ are given, the results may be obtained by simply interchanging sides and angles in the table.

* Compare Prof. Lloyd Tanner, *Messenger of Mathematics*, vol. 14, 1885, p. 153.

## A Method of Dividing the Circumference of a Circle into 360 equal parts.

By James N. Miller.

This Graduation of the Circumference of a Circle is effected by the aid of an instrument called a trisector which I contrived with the view of trisecting an angle by its assistance, but subsequently perceived that it could help to divide an angle into 5 equal angles, and recently discovered that it could contribute towards dividing the circumference of a circle into 360 equal degrees or arcs.

That instrument is outlined in the diagram (Fig. 32). It consists, as was formerly explained to this Society, of two flat or nearly flat pieces BEFGD and AHJ of peculiar forms; and may be made of metal or other material. Those two pieces are conjoined, as the blades of a pair of scissors are, by a small cylindrical pin inserted in a small cylindrical hole at C in each of them, which it fits, and round which they can be moved in either direction. A similar hole is made through the piece BEFGD at B. The centres of those holes at B and C, and also the points E and D are all in the same straight line BECD. An arm EFG projects from one of the sides of this piece. The edge EF of that arm is straight, and is perpendicular to the line BC which joins the centres of the holes at B and C. It also bisects the line BC in the point E. The line BE is therefore equal to the line EC.

The point A of the other piece AHJ, the centre of the hole at C, and its edge IJ, are all in the straight line ACIJ. The line AC, drawn from the point A to the centre at C, is exactly equal to the line BC which connects the centres at B and C. The triangle ACB is therefore isosceles.

The circumference of the circle, which it is proposed to graduate, may in the first instance be divided, as is easily done, into 3 equal arcs of 120°. Let MS be such an arc of 120°, B the centre of its circle, and BM and BS radii of the circle drawn to the extremities M and S of the arc. Also, let KL be a perpendicular to the radius BS drawn from the point K in it, K being at the same distance

from the centre at B as the point E is from the centre of the hole at B. The lines EB and KB are therefore equal.

In employing the instrument, with a view to the graduation of the circumference of the circle, a sharp cylindrical pin which fits, but not tightly, the hole at B, is inserted through that hole into the centre B of the circle, and at right angles to its plane. The point A of the piece AHJ is then moved along the radius BM until its edge IJ meets the edge EF of the arm EFG at a point, which may be termed O, on the perpendicular KL. Next, through O draw the radius BOQ, and bisect the angle CBO or NBQ with the radius BP and the angle OBS or QBS with the radius BR; and produce the line BECD till it meets the arc MS at N.

Now, as by construction, the sides BE and CE of the right-angled triangles BOE and COE are equal, and as EO is a corresponding side of both those triangles, so their corresponding angles CBO or NBQ and BCO are equal. But, as ACB is an isosceles triangle, so its angle ABC or MBN is equal to half of its exterior angle BCO. The angle ABC or MBN is therefore equal to each of the angles NBP and PBQ, they being halves of the angle NBQ, which has just been shown to be equal to the angle BCO. Thus the 3 angles MBN, NBP and PBQ are all equal.

Again, the sides BK and BE of the right-angled triangles BOK and BOE are, by construction, equal to each other; and, as these triangles have the same hypotenuse BO, so their corresponding angles OBK and OBE or NBQ are also equal to each other; and therefore their halves RBS, QBR, PBQ and NBP are equal to one another and to the angle MBN. Consequently the 5 arcs RS, QR, PQ, NP and MN by which those 5 equal angles are subtended, are all equal to each other. Each of those arcs is therefore a fifth of the arc MS of 120°, and is accordingly an arc of 24°.

Further, each of those 5 arcs may be divided into 3 equal arcs of 8°. To do this the trisector is moved round the pin at B, and the point A of the piece AHJ is moved along the radius BR until, when it is at a point A′, the edges IJ and EF, which may now be denoted by I′J′ and E′F′, meet at a point which may be termed O′ on the radius BS. Now, through the centre at C′ in the piece A′H′J′ draw the radius BC′N′, and bisect the angle N′BS with the radius BT. The arc RS is thereby divided into 3 equal arcs RN′, N′T, and TS. For, as it was proved that the angle ABC or

MBN is equal to each of the angles NBP and PBQ, so it may similarly be shown that the angle A′BC′ or RBN′ is equal to each of the angles N′BT and TBS. Consequently the 3 arcs RN′, N′T and TS, of which the arc RS of 24° consists, and which subtend those equal angles, are equal to each other. Therefore each of them is an arc of 8°.

The other 4 arcs MN, NP, PQ and QR of 24° each, may each be similarly divided into 3 arcs of 8°, and thereby the entire arc MS of 120° would be divided into 15 arcs of 8°. Each of those 15 arcs may then be easily divided into 2 arcs of 4°, which may be divided into 4 arcs of 2°, which may be further divided into 8 arcs of 1°. The entire arc MS would thereby be divided into 120 equal arcs of 1°.

The other 2 arcs of 120° may each be similarly divided into 120 arcs of 1°. In other words, the entire circumference of the circle may be thus divided or graduated into 360 equal arcs of 1°.

# Polar Loci.

By D. G. Taylor, M.A.

This paper contains two parts :—

I. *An endeavour to remove the present confusion in polar diagrams.*

II. *On the curves derived from a given curve by increasing or diminishing the vectorial angle in a constant ratio.*

## I.

1. In order to cover the whole plane of $xy$, $x$ and $y$ must both vary from $-\infty$ to $+\infty$; but the same plane is covered while $r$ varies from $-\infty$ to $+\infty$, and $\theta$ through any range $\pi$. Hence when we allow $\theta$ any larger amplitude than this, we create confusion. The curve $r = a\theta$ appears to give an infinite number of values of $r$ for each value of $\theta$, and the curve $\frac{l}{r} = 1 + e\cos\theta$ seems to give two, while the equation in each case gives only one; while for the curve $r^2 = a^2\cos 3\theta$, the region between $\frac{5\pi}{6}$ and $\pi$ appears, on considering $\frac{5\pi}{6} < \theta < \pi$, to be unoccupied, and on considering $-\frac{\pi}{6} < \theta < 0$, to be occupied.

We can remove the confusion by two simple suggestions.

(i) Confining ourselves for the moment to positive values of $r$, conceive an infinite number of planes, one above another, and each slit up from 0 to $\infty$ along the initial line. Let the under lip of each slit be joined along its length to the upper lip on the sheet just above; this produces a surface on which $\theta$ can vary from indefinitely large negative values on the lowest planes, to indefinitely large positive values on the highest.

(ii) Now considering any single plane, conceive that the radius vector, in passing through the origin, passes to the *under side of the plane*, and so is drawn *on the back* of the paper. Thus the two sides of each sheet are utilised (the upper for positive, the under for negative, values of $r$), and we have a *unifacial* surface (see

Forsyth T. F., § 165) as a proper field for the representation of the locus $f(r, \theta) = 0$.

2. Now each plane (both sides included) corresponds to variation of $\theta$ through a range $2\pi$. Hence if $f(r, \theta)$ admits the period $2\pi$ with respect to $\theta$, the curves on the different planes will be identical. This will be the case, not only for all curves algebraic in $x$ and $y$, but also for many types of transcendental curves.

Or $f(r, \theta)$ may admit some period $2m\pi$, where $m$ is an integer greater than unity. In this case the curve will begin to repeat itself after having described $m$ planes of the surface.

Again, the period of $f(r, \theta)$ may be an *irrational* multiple of $2\pi$; in which case the curve will repeat itself, but identical portions will not come under one another.

Lastly, $f(r, \theta)$ may not be periodic in $\theta$ at all.

3. In general, there will be portions of the locus on both sides of each plane; those on the upper side being described by positive, and those on the under side by negative, radii vectores. Thus in the hyperbola $\frac{l}{r} = 1 + e\cos\theta$, the more remote branch is described altogether by negative radii vectores, and is therefore to be conceived as drawn on the *back* of our paper. We may draw it in as a *dotted* line to remind ourselves of this fact.

Again, the circle $r = a\sin\theta$ is described twice while $\theta$ varies through a range $2\pi$; once with positive, and once with negative, radii vectores. It thus appears identically on both sides of the paper, and we may represent this by a dotted circle immediately inside a continuous one (Fig. 33). In each case when a curve passes through the origin, it changes to the other side of the paper, so that dotted and continuous lines are described alternately.

This mode of connection between the two sides of a plane is exactly what we would obtain from an hyperboloid of one sheet, with its generators drawn, if we were to flatten it into the plane of its principal elliptic section, and at the same time to contract that section into a pinhole. The surface would then consist of the two sides of the plane, and each generator would change to the other face of the plane in passing through the pinhole.

4. The curves $r = a\theta$, $r\theta = a$, appear in the ordinary diagram to have an indefinite number of double points. But our new convention enables us to discriminate.

($\alpha$) A *bona fide* double point, *i.e.*, one in which two branches of the locus actually meet on our surface, must satisfy the equations

$$f(r,\theta)=0,$$
$$\frac{\partial}{\partial r}f(r,\theta)=0.$$

($\beta$) But a point which satisfies the equations

$$f(r,\theta) \qquad =0,$$
$$f(r,\theta+2n\pi)=0,$$

or the equations

$$f(r,\theta) \qquad =0,$$
$$f(-r,\theta+\overline{2n+1}\pi)=0,$$

where $n$ is a positive or negative integer,

will in ordinary diagrams be mistaken for a double point. In the former case, the branches are on different planes of the surface; in the latter, they may be on the same or different planes, but are on opposite *sides* of the surface. We may call them *pseudo*-double points of the first and second kinds respectively. The apparent double points on the two spirals mentioned are *pseudo*-double points of the second kind.

## II.

1. Consider the straight line $r\sin\theta=a$. If we diminish the vectorial angle of each point in the ratio $1:m$, we obtain a Cotes' Spiral $r\sin m\theta=a$; and if we *increase* the vectorial angles in the ratio $m:1$, we obtain the curve $r\sin\dfrac{\theta}{m}=a$. The same two processes applied to the circle $r=a\sin\theta$, the parabola $r=\dfrac{4a\cos\theta}{\sin^2\theta}$, the conic $\dfrac{l}{r}=1+e\cos\theta$, the folium of Descartes $r=\dfrac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}$, the cubical parabola $r^2=\dfrac{a^2\cos\theta}{\sin^3\theta}$, the semicubical $r=\dfrac{a\cos^2\theta}{\sin^3\theta}$, or any other known curve, will likewise produce in each case two families of curves, each family with notable characteristics. Curves of the first family in each case will consist of repetitions, symmetrically placed about the origin, of a narrow (open or closed) loop. The second family will be marked by wide overlapping loops, with an abundance of *pseudo*-double points.

Conversely, by altering the vectorial angles in a constant ratio,

we are able to reduce many types of polar loci to simple algebraic curves.

2. An algebraic curve is, from the polar point of view, a locus with period $2\pi$ in $\theta$. It will in general (applying the foregoing theory) require both sides of a plane for its representation, and, when put upon our surface, will be identically repeated on each of our infinite number of sheets. When we alter the vectorial angles in a given ratio, we are simply opening or closing our surface like a fan, and the various layers of our locus become separated. When the ratio in which the angles have been altered can be expressed as the ratio of one whole number to another, the curve will repeat itself after a definite number of sheets.

Below are some of the simplest of the curves thus derived from the circle $r = a\sin\theta$.

A branch drawn in a continuous line is one which occupies the *upper* side of its sheet, and corresponds to *positive* radii vectores; while a *dotted* line denotes a branch on the *under* side of its sheet, and corresponding to *negative* radii vectores. When a branch is described in both of these ways, the continuous and dotted lines are drawn alongside one another.

It will be noticed that a curve, on passing through the origin, always *changes sides* on its sheet.

3. First, consider those produced by what we may call "contraction" of the surface. Their general equation is

$$r = a\sin m\theta$$

and we shall take $m$ an integer. In all cases there are $m$ loops above and $m$ below; but since, when $m$ is odd,

$$\sin m\theta = -\sin m(\theta + \overline{2n+1}\pi)$$

for all values of $\theta$, each point on the curve satisfies the condition for a *pseudo*-double point of the second kind, so that each under loop will be covered by an upper; while, for $m$ even, they will be found in separate regions. The enveloping circle $r = a$ is drawn in each case, and the loops are numbered in the order of their description.

Figures are drawn for $m = 1, 2, 3, 4$. (Figures 34, 35, 36, 37.)

4. The general equation of curves derived from the circle

$$r = a\sin\theta$$

by what we may call "extension" of the surface is

$$r = a\sin\frac{\theta}{m}.$$

As before, we consider $m$ integral. The complete variation of $r$ takes place while $\theta$ goes from 0 to $2m\pi$; there will thus be $m$ sheets before the curve begins to repeat.

Figures are drawn for $m=2$, 3, 4. (Figures 38, 39, 40.) The following short notes indicate the order in which the different branches are described; and each branch is numbered in the figures according to the sheet in which it lies.

(i) $r = a\sin\dfrac{\theta}{2}$ (Fig. 38).

$0<\theta<2\pi$: continuous line OAO, upper surface of first sheet;
$2\pi<\theta<4\pi$: dotted line OBO, under surface of second sheet.

(ii) $r = a\sin\dfrac{\theta}{3}$ (Fig. 39).

$0<\theta<2\pi$: OAB continuous, first sheet;
$2\pi<\theta<3\pi$: BCO continuous, second sheet;
$3\pi<\theta<4\pi$: OCA dotted, second sheet;
$4\pi<\theta<6\pi$: ABO dotted, third sheet.

(iii) $r = a\sin\dfrac{\theta}{4}$ (Fig. 40).

$0<\theta<2\pi$: OAC continuous, first sheet;
$2\pi<\theta<4\pi$: CAO continuous, second sheet;
$4\pi<\theta<6\pi$: OBD dotted, third sheet;
$6\pi<\theta<8\pi$: DBO dotted, fourth sheet.

Similarly for higher values of $m$.

It is useful to note that

$$\tan\phi = \frac{rd\theta}{dr} = m\tan\frac{\theta}{m}.$$

Since, for $m$ odd, $\sin\dfrac{\theta}{m} = -\sin\dfrac{\theta+\overline{2n+1}\pi}{m}$ for $n=\dfrac{m-1}{2}$ and all values of $\theta$, the same curve, just as in § 3, will be described by negative radii vectores as by positive; but not so for $m$ even. This is illustrated by the examples drawn.

The point to note is, that those curves, becoming more and more involved as $m$ increases, are simple "extensions" of the circle; and, finally, that without the conception of a many-sheeted surface, their diagrams would be a mass of confusion.

*Fifth Meeting, 10th March* 1905.

Mr W. L. Thomson, President, in the Chair.

## Some Proofs of Newton's Theorem on Sums of Powers of Roots.

By R. F. Muirhead, M.A., D.Sc.

This well-known theorem, published in Newton's *Arithmetica Universalis*, may be formulized thus:—

$$s_m - p_1 s_{m-1} + p_2 s_{m-2} \dots \mp p_{m-1} s_1 \pm m p_m = 0 \quad \text{-} \quad (1)$$

where $s_m \equiv$ the sum of the $m$th powers of the roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} \dots \pm p_n = 0. \quad \text{- - -} \quad (2)$$

Of course, when $m > n$, certain coefficients $p_{n+1}, p_{n+2}, \dots p_m$ will be zero, and the last term that does not vanish will be $\pm p_n s_{m-n}$.

Several elementary proofs of this theorem which have not hitherto been printed have accumulated in my notes during past years. They are elementary in the sense that they do not involve the use of infinite series. The present communication has been suggested to me by a perusal of the note by Mr Tweedie in last session's *Proceedings*, where a brief and suggestive proof of the theorem is given.

Of the proofs given in the present paper, the first is one for which I am indebted to Mr John Dougall, who showed it to me several years ago. The second, perhaps, is the most direct. The third is a modification of the line of proof given in Chapter XVIII. of Chrystal's *Algebra*.

### First Proof.

Denote by F$(n)$ the expression

$$(s_m - p_1 s_{m-1} + p_2 s_{m-2} - \dots \pm p_{m-1} s_1 \mp m p_m)_n$$

where $s_r \equiv$ the sum of the $r$th powers of the $n$ letters $a_1, a_2, \dots a_n$

and $p_r \equiv$ " " " " products of $a_1, a_2, \dots a_n$ taken $r$ at a time.

Then for the case of $n = m$, the quantities $a_1, a_2, \dots$ are the roots of the $m$-ic

$$x^m - p_1 x^{m-1} + p_2 x^{m-2} - \dots \pm p_{m-1} x \mp p_m = 0. \quad \text{- -} \quad (3)$$

Substituting successively $a_1, a_2, \ldots a_m$ for $x$ in (3) and adding the results, we get

$$F(m) = 0. \quad - \quad - \quad - \quad (4)$$

Now suppose $n = m + 1$, then if any one of the quantities $a_1, a_2, \ldots a_n$ vanish, $F(m+1)$ will reduce to $F\ (m)$ and will therefore vanish. Hence that particular $a$ must be a factor of $F(m+1)$. Similarly we see that each of the other letters $a$ must be a factor of $F(m+1)$. This is only possible if $F(m+1)$ is identically zero, since it is only of dimensions $m$ in the $a$'s.

Thus $F(m+1) = 0$ identically.

In a similar manner we can show that $F(m+2)$ is identically zero, since it reduces to $F(m+1)$, and therefore to zero when one of its $m+2$ $a$'s vanishes. And so on, by mathematical induction.

Thus when $n \not< m$ we have $F(n) = 0$.

The proof when $n < m$ is easy, by taking an $n$-*ic* and multiplying by $x^{m-n}$ and then substituting $a_1, a_2, \ldots a_n$ successively for $x$ in it, and adding: but here, of course, $p_{n+1}, p_{n+2}, \ldots p_m$ are all zero, and $F(n)$ reduces to

$$s_m - p_1 s_{m-1} + p_2 s_{m-2} \cdots \pm p_n s_{m-n}.$$

## Second Proof.

Given $n$ letters $a, \beta, \gamma, \ldots$ we have

(5)...$\Sigma a^p \Sigma(a\beta\gamma \ldots \text{to } q \text{ factors}) = \Sigma(a^{p+1}\beta\gamma\ldots) + \Sigma(a^p\beta\gamma\ldots \text{to } \overline{q+1} \text{ factors})$

excepting when $p = 1$, in which case we have

(6)...$\Sigma a\ \Sigma(a\beta\gamma\ldots \text{to } q \text{ factors}) = \Sigma(a^2\beta\gamma..) + (q+1)\Sigma(a\beta..\text{to } \overline{q+1} \text{ factors})$.

Hence we deduce

$$\begin{aligned}
&\Sigma a^m - \Sigma a \Sigma a^{m-1} + \Sigma a\beta \Sigma a^{m-2} - \ldots \pm \Sigma(a\beta\gamma \ldots \text{to } m-1 \text{ factors})\ \Sigma a \\
&= \Sigma a^m - (\Sigma a^m + \Sigma a^{m-1}\beta) + (\Sigma a^{m-1}\beta + \Sigma a^{m-2}\beta\gamma) - \ldots \\
&\qquad \mp \{\Sigma(a^3\beta\gamma\ldots) + \Sigma(a^2\beta\gamma\ldots)\} \\
&\qquad \pm \{\Sigma(a^2\beta\gamma\ldots) + m\Sigma(a\beta\gamma \ldots \text{to } m \text{ factors})\} \\
&= \pm m\Sigma(a\beta \ldots \text{to } m \text{ factors}).
\end{aligned}$$

Thus, using the previous notation, we have

$$s_m - p_1 s_{m-1} + p_2 s_{m-2} \dots \pm p_{m-1} s_1 \mp m p_m = 0$$

which is Newton's Theorem, excepting that $p_1$, $p_2$, ... are here *defined* as symmetric functions, instead of as coefficients of an equation.

This proof applies equally for all positive integral values of $m$; but, of course, if $m > n$, the values of $p_{n+1}$, $p_{n+2}$, ... $p_m$ will be zero.

## Third Proof.

Let $p_r$ denote the sum of the products of $n$ letters $\alpha$, $\beta$, $\gamma$, ... taken $r$ at a time, and $a_r$ denote the same function of the $n-1$ letters $\beta$, $\gamma$, $\delta$, ..., and $b_r$ that of the $n-1$ letters $\alpha$, $\gamma$, $\delta$, ..., and so on. Then it is obvious from first principles that

$$\begin{aligned} p_r &= \alpha a_{r-1} + a_r \\ p_{r-1} &= \alpha a_{r-2} + a_{r-1} \\ &\dots\dots\dots\dots \\ p_2 &= \alpha a_1 \quad + a_2 \\ p_1 &= \alpha \qquad + a_1. \end{aligned}$$

Multiplying these equations respectively by

$$1, \ -\alpha, \ \alpha^2, \ -\alpha^3, \dots \pm \alpha^{r-2}, \ \mp \alpha^{r-1},$$

and adding, we get

$$p_r - \alpha p_{r-1} + \alpha^2 p_{r-2} \dots\dots \mp \alpha^{r-1} p_1 = a_r \mp \alpha^r \quad - \quad - \quad (7)$$

Similarly we can show that

$$\begin{aligned} p_r - \beta p_{r-1} + \beta^2 p_{r-2} \dots\dots \mp \beta^{r-1} p_1 &= b_r \mp \beta^r \\ p_r - \gamma p_{r-1} + \gamma^2 p_{r-2} \dots\dots \mp \gamma^{r-1} p_1 &= c_r \mp \gamma^r \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

Summing these identities, we get

$$n p_r - s_1 p_{r-1} + s_2 p_{r-2} \dots\dots \mp s_{r-1} p_1 = \overline{n - r}\, p_r \mp s_r,$$

the factor $n - r$ being due to the fact that each term of $p_r$ occurs in $n - r$ of the quantities $a_r$, $b_r$, etc., viz., in those which do not exclude any of the $r$ letters of which that term is formed.

Hence, transposing, we have

$$rp_r - s_1 p_{r-1} + s_2 p_{r-2} - \ldots\ldots \mp s_{r-1} p_1 \pm s_r = 0.$$

Thus Newton's Theorem is proved for all positive integral values of $r$. Of course when $r > n$, certain $p$'s will be zero.

This method of proof is in essential respects analogous to that given in Chrystal's *Algebra*, Chapter XVIII., from which, however, it differs principally with respect to its starting point, *i.e.*, in starting from the elementary symmetric functions instead of from an equation. It is perhaps worthy of note that, as put here, the investigation enables us to deduce as a corollary the fact that $-p_1$, $p_2$, $-p_3$, etc., are the coefficients of the $n$-*ic* whose roots are $a$, $\beta$, $\gamma$, ... .

The equation (7) may be written

$$a^r - p_1 a^{r-1} + p_2 a^{r-2} \ldots\ldots \pm p_r = \pm a_r.$$

Putting $r = n$ and observing that in this case $a_r = a_n = 0$,

we get $$a^n - p_1 a^{n-1} + p_2 a^{n-2} \ldots\ldots \pm p_n = 0.$$

Thus the equation $x^n - p_1 x^{n-1} \ldots \pm p_n = 0$ is satisfied by each of the $n$ quantities $a$, $\beta$, $\gamma$, ... ; *i.e.*, it is the equation of which these are the roots.

---

It may be of interest to generalise Newton's Formula to a certain extent, by using the method of the second proof.

The equation (5) may be seen to hold good for all values of $p$, whether positive, negative, or even fractional, provided we agree that in each $\Sigma$ on the right hand side of (5) one letter is to have an index attached, which may be 0 or 1 in certain cases, and that

$\Sigma a^1(\beta\gamma\delta \ldots$ to $q$ factors) is the same as $(q+1)\Sigma(a\beta \ldots\ldots$ to $q+1$ factors)

and that

$\Sigma a^0(\beta\gamma\delta \ldots$ to $q$ factors) is the same as $(n-q)\Sigma(a\beta\gamma \ldots$ to $q$ factors)

so that $\Sigma a^0$ is the same as $n$. The exceptional case of $p = 1$ (equation 6) will then be included in the general formula (5).

To illustrate the modified notation, let us take the case of four quantities $\alpha$, $\beta$, $\gamma$, $\delta$. Here we have

$$\Sigma\alpha^1\beta\gamma \equiv \alpha^1\beta\gamma + \alpha^1\beta\delta + \alpha^1\gamma\delta + \beta^1\alpha\gamma + \beta^1\alpha\delta + \beta^1\gamma\delta + \gamma^1\alpha\beta + \gamma^1\alpha\delta + \gamma^1\beta\delta$$
$$+ \delta^1\alpha\beta + \delta^1\alpha\gamma + \delta^1\beta\gamma$$
$$= 3(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta),$$
$$\Sigma\alpha^0\beta\gamma \equiv \alpha^0\beta\gamma + \alpha^0\beta\delta + \alpha^0\gamma\delta + \beta^0\alpha\gamma + \ldots\ldots$$
$$= 2(\beta\gamma + \beta\delta + \gamma\delta + \alpha\beta + \alpha\gamma + \alpha\delta)$$

and $\Sigma\alpha^0 = \alpha^0 + \beta^0 + \gamma^0 + \delta^0 = 4.$

In the identity (5) read in this way, let us substitute for $p$ the values $m$, $m-1$, $m-2, \ldots m-r+1$, successively, multiply these equations by $+1$ and by $-1$ alternately, and add the results together. Noting that each term is of dimensions $m$, we have

$$\Sigma\alpha^m - (-1)^r\Sigma(\alpha^{m-r}\beta\gamma\ldots)$$
$$= \Sigma\alpha\Sigma\alpha^{m-1} - \Sigma\alpha\beta\Sigma\alpha^{m-2} + \Sigma\alpha\beta\gamma\Sigma\alpha^{m-3} - \ldots - (-)^r\Sigma(\alpha\beta\ldots)\Sigma\alpha^{m-r}.$$

Hence we have

$$\Sigma\alpha^m - \Sigma\alpha\Sigma\alpha^{m-1} + \Sigma\alpha\beta\Sigma\alpha^{m-2} - \ldots + (-1)^r\Sigma(\alpha\beta\ldots)\Sigma\alpha^{m-r}$$
$$= (-1)^r\Sigma(\alpha^{m-r}\beta\gamma\ldots) \quad - \quad - \quad - \quad (8)$$

A further generalisation is got by subtracting this from the corresponding formula with $s$ substituted for $r$.

This gives

$$p_{r+1}\,s_{m-r-1} - p_{r+2}\,s_{m-r-2} + p_{r+3}\,s_{m-r-3} - \ldots - (-1)^{s-r}p_s\,s_{m-s}$$
$$= \Sigma(\alpha^{m-r}\beta\gamma\ldots) - (-1)^{s-r}\Sigma(\alpha^{m-s}\beta\gamma\ldots) \quad - \quad - \quad (9)$$

Now note the special case when $s = r+2$. Equation (9) then becomes

$$p_{r+1}\,s_{m-r-1} - p_{r+2}\,s_{m-r-2} = \Sigma(\alpha^{m-r}\beta\gamma\ldots) - \Sigma(\alpha^{m-r-2}\beta\gamma\ldots) \quad - \quad (10)$$

This equation (10) could also be easily deduced directly from first principles.

# A proof of Waring's Expression for $\Sigma a^r$ in terms of the Coefficients of an Equation.

By R. F. Muirhead, M.A., D.Sc.

While Newton's Theorem on the Sums of Powers of the Roots of an equation furnishes a set of lineo-linear equations connecting the quantities $s_1$, $s_2$, $s_3$, ... and the quantities $p_1$, $p_2$, $p_3$, ... Waring gives the solution of these equations by which the $s$'s are expressed in terms of the $p$'s.

The general formula for $s_r$ given by Waring, both in his *Meditationes Algebraicae* and in his *Miscellanea Analytica* is sometimes named after Albert Girard, who a century earlier, in his *Invention Nouvelle en l'Algèbre* gave the formulas for the sums of the squares, cubes and fourth powers; but as this mathematician gave no hint as to the form of the general formula, and perhaps even did not suspect the possibility of a general formula, it seems to me that if any name is to be associated with the formula, that name ought to be Waring's.

Waring gives a succinct proof by Mathematical Induction. This, though quite complete, has of course the disadvantage of requiring a knowledge of the formula to start with. A variety of other proofs have been given, of which the simplest are those which, like that indicated in Burnside and Panton's *Theory of Equations*, § 133, Ex. 8, use expansions by the Multinomial Theorem, and equate coefficients of like powers.

The proof here given is of that character, but it is perhaps unique in being *elementary* in the sense that it does not use infinite series.

Let $a$, $\beta$, $\gamma$, ... $a_n$ be the roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} \dots \pm p_n = 0\,;$$

and let $s_r$ denote the sum $a^r + \beta^r + \gamma^r + \dots + a_n{}^r$.

Then we have the identity

$$(1 + ax)(1 + \beta x)(1 + \gamma x) \dots\dots\dots = 1 + p_1 x + p_2 x^2 + \dots + p_n x^n. \quad (1)$$

Hence

$$(1 + ax)^m (1 + \beta x)^m (1 + \gamma x)^m \dots . = (1 + p_1 x + p_2 x^2 + \dots + p_n x^n)^m. \quad (2)$$

But $(1 + ax)^m = 1 + \binom{m}{1} ax + \binom{m}{2} a^2 x^2 + \dots + \binom{m}{r} a^r x^r + \dots + \binom{m}{m} a^m x^m.$

Expanding both sides of (2) we find the general term on the right to be

$$\frac{m!\,p_1^{r_1}p_2^{r_2}p_3^{r_3}\ldots p_n^{r_n}\,.\,x^{r_1+2r_2+\ldots+nr_n}}{r_0!\,r_1!\,r_2!\ldots r_n!}$$

where $r_0, r_1, \ldots r_n$ are integers or zeros such that $r_0+r_1+r_2+\ldots+r_n=m$.

The general term on the left is

$$\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}\alpha^a\beta^b\gamma^c\ldots a_n^{a_n}\,.\,x^{a+b+c\ldots+a_n}$$

where $a, b, c, \ldots$ may have any values from 0 up to $m$ inclusive.

Thus on the right, the coefficient of $x^r$ is

$$\Sigma\frac{m!\,p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}}{r_0!\,r_1!\ldots r_n!}$$

where the summation extends over all values of $r_0, r_1, \ldots$ for which

$$r_0+r_1+\ldots+r_n=m \quad\text{and}\quad r_1+2r_2+3r_3+\ldots+nr_n=r.$$

On the left, the coefficient of $x^r$ is

$$\Sigma\left\{\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}\alpha^a\beta^b\ldots a_n^{a_n}\right\}$$

where the summation extends over all possible sets of positive integral or zero values of $a, b, c, \ldots a_n$ for which $a+b+c\ldots+a_n=r$.

Of these sets, some will differ only as to the *order* in which the indices $a$, $b$ and $c$ occur. In order to group together such sets, let us denote by $(a, b, c, \ldots a_n)$ the symmetrical function $\Sigma(\alpha^a\beta^b\ldots a_n^{a_n})$ where the summation extends over all *different* products which can be got by permuting the *indices* while $\alpha, \beta, \ldots$ are kept in one definite order.

The coefficient of $x^r$ on the left can then be written

$$\Sigma\left\{\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}(a, b, c, \ldots a_n)\right\}.$$

Equating coefficients of $x^r$ on the right and on the left, we get

$$(3)\quad \Sigma\left\{\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}(a, b, c, \ldots a_n)\right\}$$
$$=\Sigma\frac{m(m-1)(m-2)\ldots(m+1-r_1-r_2-\ldots-r_n)}{r_1!\,r_2!\ldots\ldots r_n!}p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}$$

where $a+b+\ldots+a_n=r$, and $r_1+2r_2+3r_3+\ldots+nr_n=r$, and on the left the order of $a, b, c, \ldots a_n$ is indifferent.

With respect to $m$ each side of this identity is a rational integral algebraic function of degree $r$, the form of which is independent of $m$ provided $m$ is large enough. Hence we may equate coefficients of powers of $m$.

The coefficient of $m$ on the left arises from such terms as have only *one* of the quantities $a, b, c, \ldots$ different from zero, that one being $= r$ and it is therefore

$$= \frac{(-1)(-2)\ldots(-r+1)}{1.2 \ldots\ldots\ldots\ldots r}(r, 0, 0, 0, \ldots),$$

which may be written $\qquad (-1)^{r-1}\frac{1}{r}\Sigma a^r.$

The coefficient of $m$ on the right is

$$\Sigma \frac{(-1)(-2)\ldots(-r_1-r_2-r\ldots-r_n+1)}{r_1!\,r_2!\ldots r_n!} p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}.$$

Thus we have Waring's Formula

$$(4) \quad \Sigma a^r = r\Sigma \frac{(r_1+r_2+\ldots+r_n-1)!\,p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}(-1)^{\Sigma r_1 - r}}{r_1!\,r_2!\,r_3!\ldots r_n!}$$

$$\text{or} \quad \Sigma(-a^r) = r\Sigma \frac{(r_1+r_2+\ldots+r_n-1)!\,(-p_1)^{r_1}(-p_2)^{r_2}\ldots(-p_n)^{r_n}}{r_1!\,r_2!\ldots r_n!}$$

where $r_1, r_2, \ldots r_n$ have all possible positive integer or zero values making

$$r_1 + 2r_2 + 3r_3 \ldots nr_n = r.$$

---

It may be of interest to note the more complicated formula which arises when we equate the coefficients of $m^k$ on the right and left of (3), $k$ being a positive integer not greater than $r$. It may be written

$$(-1)^{r-k}\Sigma\frac{[a, b, c, \ldots a_n;\, k](a, b, c, \ldots a_a)}{a!\,b!\,c!\ldots\ldots\ldots\ldots a_n!}$$

$$= \Sigma \frac{(-1)^{p-k}[p;\, k]\,p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}}{r_1!\,r_2!\ldots\ldots r_n!} \quad \cdot \quad \cdot \quad (5)$$

where $[a, b, c, \ldots a_n;\, k]$ denotes the sum of all products of the numbers $1, 2, \ldots\ldots a-1$; $1, 2, \ldots\ldots b-1$; $\ldots\ldots$; $1, 2, \ldots\ldots a_n - 1$, taken $a+b+\ldots+a_n-k$ at a time, the value of $[a, b, c, \ldots a_n;\, k]$ being

reckoned $=1$ if $a+b+\dots+a_n-k$ is zero, and $=0$ if the latter is negative; and $p$ denotes the sum $r_1+r_2+r_3+\dots+r_n$; and as before $a$, $b$, etc., have any positive integral or zero values making $a+b+c+\dots+a_n=r$, and $r_1$, $r_2$, etc., have any positive integral or zero values making $r_1+2r_2+3r_3+\dots+nr_n=r$.

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It may be useful to collect the formulae by which the elementary symmetrical functions $p_1$, $p_2$, etc., the sums of powers $s_1$, $s_2$, etc., and the sums of homogeneous products $H_1$, $H_2$, etc., are expressed in terms of one another. In addition to Waring's Formula, we have five others, as follows:—

$$(6)\qquad s_r = r\sum\frac{(r_1+r_2+\dots+r_m-1)!(-1)^{r_1+r_2+\dots+r_m-1}H_1^{r_1}H_2^{r_2}\dots H_m^{r_m}}{r_1!\,r_2!\,r_3!\dots\dots\dots r_m!}$$

$$(7)\qquad p_r=(-1)^r\sum\frac{(-s_1)^{r_1}(-s_2)^{r_2}\dots(-s_m)^{r_m}}{(r_1!\,r_2!\dots r_m!)(2^{r_2}3^{r_3}\dots m^{r_m})}$$

$$(8)\qquad H_r=\sum\frac{s_1^{r_1}s_2^{r_2}\dots s_m^{r_m}}{(r_1!\,r_2!\dots r_m!)(2^{r_2}3^{r_3}\dots m^{r_m})}$$

$$(9)\qquad H_r=(-1)^r\sum\frac{(r_1+r_2+\dots+r_n)!(-p_1)^{r_1}(-p_2)^{r_2}\dots(-p_n)^{r_n}}{r_1!\,r_3!\dots\dots r_n!}$$

$$(10)\qquad p_r=(-1)^r\sum\frac{(r_1+r_2+\dots+r_m)!(-H_1)^{r_1}(-H_2)^{r_2}\dots(-H_m)^{r_m}}{r_1!\,r_2!\,r_3!\dots\dots r_m!}$$

where in each case $r_1$, $r_2$, ... are to have all possible positive integral or zero values for which $r_1+2r_2+3r_3+\dots=r$.

It is to be noted that the series of $p$'s ends with $p_n$, while the series of $s$'s and of H's do not end. Note also that many writers use $(-1)^r p_r$ to denote what is here denoted by $p_r$.

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## Determination of the radii of the circles which touch three given circles.

By Alex. Holm, M.A.

Let the points A, B, C be the centres of three given circles, whose radii are $a$, $b$, $c$; and let $d$, $e$, $f$ be the distances BC, CA, AB between the centres (Fig. 41). It is required to find the radii of the circles which touch the circles A, B, C.

1. Suppose that O is the centre and $x$ the radius of a circle which touches the circles A, B, C all externally at the points P, Q, R.

Then OA, OB, OC pass through the points of contact.

$$\therefore \quad \mathrm{OA} = x + a, \quad \mathrm{OB} = x + b, \quad \mathrm{OC} = x + c.$$

Let QR, the straight line joining the points of contact of the tangent circle O with the circles B, C, cut those circles again at Q′, R′, and meet BC in S. Then S is the direct centre of similitude of the circles B, C.

$\therefore$ CQ′ is parallel to BQ and BR′ to CR.

$$\therefore \quad \frac{\mathrm{QR}}{\mathrm{QQ'}} = \frac{\mathrm{OR}}{\mathrm{OC}} = \frac{x}{x+c} \quad \text{and} \quad \frac{\mathrm{QR}}{\mathrm{R'R}} = \frac{\mathrm{OQ}}{\mathrm{OB}} = \frac{x}{x+b}.$$

$$\therefore \quad \frac{\mathrm{QR}^2}{\mathrm{QQ'.R'R}} = \frac{x^2}{(x+b)(x+c)}.$$

Now QQ′. R′R is constant for any secant through the centre of similitude S.

Hence if $l$ is the length of the direct common tangent TT′ passing through S,

$$\mathrm{QQ'.R'R} = \mathrm{TT'}^2 = l^2 = d^2 - (b-c)^2.$$

Also if $m$ and $n$ are the lengths of the direct common tangents to circles C, A, and A, B,

$$m^2 = e^2 - (c-a)^2, \quad n^2 = f^2 - (a-b)^2.$$

Then, denoting the sides of triangle PQR by $p$, $q$, $r$, we have

$$p^2 = \frac{l^2x^2}{(x+b)(x+c)}, \quad q^2 = \frac{m^2x^2}{(x+c)(x+a)}, \quad r^2 = \frac{n^2x^2}{(x+a)(x+b)}.$$

Now $x$ is the radius of the circumcircle of triangle PQR

$$\therefore \quad \frac{pqr}{4\Delta_{pqr}} = x .$$

$$\therefore \quad \frac{16\Delta^2_{pqr}}{p^2q^2r^2} = \frac{1}{x^2} .$$

$$\frac{2q^2r^2 + 2r^2p^2 + 2p^2q^2 - p^4 - q^4 - r}{p^2q^2r^2} = \frac{1}{x^2} .$$

$$\therefore \quad \frac{2}{p} + \frac{2}{q} + \frac{2}{r^2} - \frac{p^2}{q^2r^2} - \frac{q^2}{r^2p^2} - \frac{r^2}{p^2q^2} = \frac{1}{x^2} .$$

Substitute the above values of $p^2$, $q^2$, $r^2$.

$$\therefore \quad \frac{2(x+b)(x+c)}{l^2} + \frac{2(x+c)(x+a)}{m^2} + \frac{2(x+a)(x+b)}{n^2}$$

$$- \frac{l^2}{m^2n^2}(x+a)^2 - \frac{m^2}{n^2l^2}(x+b)^2 - \frac{n^2}{l^2m^2}(x+c)^2 = 1.$$

$$\therefore \quad (2m^2n^2 + 2n^2l^2 + 2l^2m^2 - l^4 - m^4 - n^4)x^2$$

$$+ 2\{(b+c)m^2n^2 + (c+a)n^2l^2 + (a+b)l^2m^2 - al^4 - bm^4 - cn^4\}x$$

$$+ (2bcm^2n^2 + 2can^2l^2 + 2abl^2m^2 - a^2l^4 - b^2m^4 - c^2n^4 - l^2m^2n^2) = 0,$$

which is a quadratic equation to find $x$.

2. Solving the quadratic, the expression under the square root is

$$4\{(b+c)m^2n^2 + (c+a)n^2l^2 + (a+b)l^2m^2 - al^4 - bm^4 - cn^4\}^2$$

$$- 4(2m^2n^2 + 2n^2l^2 + 2l^2m^2 - l^4 - m^4 - n^4)$$

$$(2bcm^2n^2 + 2can^2l^2 + 2abl^2m^2 - a^2l^4 - b^2m^4 - c^2n^4 - l^2m^2n^2).$$

This expression is of the tenth degree, and can be resolved into ten linear factors.

If we put $l^2$ or $m^2$ or $n^2 = 0$, the expression vanishes;

$$\therefore \quad d^2 - (b-c)^2,\ e^2 - (c-a)^2,\ f^2 - (a-b)^2 \text{ are factors.}$$

Again, if $d$ were equal to $e+f$, the centres of the three given circles would be in a straight line, which would be an axis of symmetry for each circle (Figs. 42, 43). Hence the two circles, which touch the given circles all externally (or all internally), would have equal radii.

Thus the two roots of the quadratic would be equal.

Therefore the expression under the square root must vanish when $d = e + f$, or when $d^2 = (e+f)^2$, since only even powers of $d$ occur in it.

Therefore $d+e+f$ and $-d+e+f$ are factors.

Similarly $d-e+f$ and $d+e-f$ are factors.

Hence the expression

$$\equiv A(d+b-c)(d-b+c)(e+c-a)(e-c+a)$$
$$(f+a-b)(f-a+b)(d+e+f)(-d+e+f)(d-e+f)(d+e-f),$$

where A is some constant coefficient.

Putting

$a=0$, $b=0$, $c=0$, $d=1$, $e=1$, $f=1$, and thus $l^2=1$, $m^2=1$, $n^2=1$,

we have $12 = 3A$, so that $A=4$.

Now $(d+e+f)(-d+e+f)(d-e+f)(d+e-f) = 16\Delta^2_{def}$,

where $\Delta_{def}$ is the area of the triangle whose sides are $d$, $e$, $f$.

Therefore the expression under the square root is $64l^2m^2n^2\Delta^2_{def}$, which is a remarkable form for it.

3. Thus the roots of the quadratic equation are

$$x = \frac{al^4 + bm^4 + cn^4 - (b+c)m^2n^2 - (c+a)n^2l^2 - (a+b)l^2m^2 \pm 4lmn\Delta_{def}}{2m^2n^2 + 2n^2l^2 + 2l^2m^2 - l^4 - m^4 - n^4},$$

where $l^2 = d^2 - (b-c)^2$, $m^2 = e^2 - (c-a)^2$, $n^2 = f^2 - (a-b)^2$.

This gives the radius of a tangent circle exterior to all the given circles.

4. To deduce the radii of the other tangent circles.

(1) For a tangent circle enveloping all the given circles,

$$OA = x - a, \quad OB = x - b, \quad OC = x - c.$$

Hence the radius $x$ could be obtained by changing the signs of $a$, $b$, $c$ in § 3.

Now those changes would leave $l$, $m$, $n$ unaltered, and as $\Delta_{def}$ remains constant, $x$ would be changed in sign only.

Therefore according as the roots in § 3 are real and

(i) both positive, (ii) one positive, the other negative, (iii) both negative, there will be

(i) two exterior tangent circles,
(ii) one exterior, and one enveloping tangent circle,
(iii) two enveloping tangent circles.

Thus when the roots are real, there is a pair of tangent circles each of which has the given circles all on the same side of it.

(2) For a tangent circle enveloping circle A, but exterior to circles B, C,

$$OA = x - a, \quad OB = x + b, \quad OC = x + c.$$

Hence the radius $x$ can be obtained by changing the sign of $a$ in § 3.

Those new roots with their signs changed belong to a tangent circle exterior to circle A, but enveloping circles B, C.

For here $\qquad OA = x + a, \quad OB = x - b, \quad OC = x - c,$

so that $a$, $b$, $c$ have now opposite signs to the preceding.

Therefore according as the new roots are real and

(i) both positive, (ii) one positive, the other negative, (iii) both negative, there will be

(i) two tangent circles enveloping circle A, but exterior to circles B, C,

(ii) one tangent circle „ „ „ „ „ „ „ „

and one tangent circle exterior to circle A, but enveloping circles B, C,

(iii) two tangent circles „ „ „ „ „ „ „ „.

Therefore when those new roots are real, there is a pair of tangent circles each of which has circle A on one side and circles B, C on the opposite side of it.

(3) To obtain the radii of the pair of tangent circles which have circle B on one side, and circles C, A on the opposite side, change the sign of $b$ in § 3.

(4) To find the radii of the pair of tangent circles having circle C on one side, and circles A, B on the opposite side, make $c$ negative in § 3.

Thus in general there are four pairs of tangent circles.

## Historical Note.

5. In Leybourn's Mathematical Questions from the *Ladies' Diary*, Vol. IV., pp. 270–275, there is a solution of the proposed problem by Binet, taken from the *Journal de l'Ecole Polytechnique*, 17 Cahier, t. x. In the relation connecting the lengths of the six straight lines

joining four points in a plane, Binet puts $x+a$, $x+b$, $x+c$ for the lengths of three lines that are concurrent, and $d$, $e$, $f$ for the lengths of the other three lines; $a$, $b$, $c$ being the radii of the three given circles, $d$, $e$, $f$ the distances between their centres, and $x$ the radius of a circle touching the given circles all externally.

After a long reduction a quadratic equation in $x$ with very complex coefficients is obtained, and no attempt is made to solve the quadratic.

We learn from Pappus that the problem "To describe a circle to touch three given circles" was the chief proposition in a lost work of Apollonius of Perga on Tangencies (c. 210 B.C.).*

Franciscus Vieta solved this problem at the close of the 16th century,† and since then it has occupied the attention of many mathematicians. In 1814 Gergonne, in the *Annales de Mathématiques*, t. vi., p. 439, gave a solution based on the theory of Radical Axis, Similitude, Pole and Polar. Gergonne's construction can also be applied, when one or two of the given circles become points or straight lines, and it seems to be the most general geometrical construction yet published.‡ For an enumeration of the different positions of the three given circles relatively to one another, with the number of possible tangent circles in each case, see an article by R. F. Muirhead "On the Number and Nature of the Solutions of the Apollonian Contact Problem" (*Proc. Edin. Math. Soc.*, Vol. XIV., pp. 135–147).

* Cf. *Hutton's Math. Dict.* pp. 129, 130, or *Leslie's Geometry*, 1811, pp. 434–437.

† For Vieta's solution see *Leybourn's Math. Quest.*, Vol. IV., pp. 262–264.

‡ Cf. *Chasles' Géométrie Supérieure*, XII. éd., pp. 498–501.

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## On the orthoptic locus of the semi-cubical parabola.

By A. G. Burgess, M.A.

*Sixth Meeting, 12th May* 1905.

Mr W. L. THOMSON, President, in the Chair.

## Bibliography of the Envelope of the Wallace Line (the three-cusped Hypocycloid).

By J. S. MACKAY, M.A., LL.D.

This bibliographical note was drawn up to accompany Mr Collignon's memoir *Recherches sur l'Enveloppe des Pédales des divers points d'une Circonférence par rapport à un triangle inscrit*, printed in this volume, p. 2–34; and if I had remembered (as I ought to have done) the very full bibliography given in *L'Intermédiaire des Mathématiciens* (Vol. 3, p. 166–168, 1896) by Mr Brocard and others, I should not have commenced it. The result, however, has been that several articles on this particular curve, not noted in the *Intermédiaire*, have been discovered, and I have thought it worth while to print the information thus gained.

Reference should also be made to the *Intermédiaire*, Vol. 1, p. 13–15, 159, 174–176 (1894), Vol. 3, p. 141–143 (1896), Vol. 5, p. 8–9 (1898), Vol. 8, p. 265–266 (1901).

The Wallace line was discovered about 1799 or 1800. See Leybourn's *Mathematical Repository* (old series), Vol. 2, p. 111.

The discovery of this line is very frequently and erroneously attributed to Robert Simson of Glasgow. The reasons for attributing the discovery of the line to Wallace and not to Simson will be found in an article in the *Proceedings of the Edinburgh Mathematical Society*, Vol. 9, p. 83–91 (1891).

1856–1866

In 1857 there appeared in Crelle's *Journal*, Vol. 53, p. 231–237, an article by Jacob Steiner, *Über eine besondere Curve dritter Klasse (und vierten Grades)*. It had been read at the Berlin Academy of Sciences on the 7th of January 1856. Steiner merely enunciates properties of the curve, the three-cusped hypocycloid, without giving demonstrations or diagrams.

This article has been republished in Steiner's *Gesammelte Werke*, Vol. 2, p. 641–647 (1882), and an abridgment of it in English will be found in *Mathematical Questions with their Solutions from the "Educational Times,"* which is generally referred to as E. T. R. (*Educational Times Reprint*), Vol. 3, p. 97–100 (1865).

Also in 1857 there appeared in Crelle's *Journal*, Vol. 54, p. 31–47, an article by H. Schröter of Breslau, *Über die Erzeugnisse krummer projectivischer Gebilde.* "Schröter noticed that the three-cusped quartic of Steiner was also the envelope of the connector of corresponding points of two anharmonically corresponding systems, one on a circle, the other on the line at infinity; and hence he was led to interesting generalisations."

In the *Lady's and Gentleman's Diary* for 1860, p. 72, the prize question proposed for solution by "Petrarch" is:

*If an hypocycloid has the radius of its describing circle one-third of that of its base, being thus composed of three branches, and a straight line equal to twice the diameter of the small circle be placed with its extremities on two of the branches, it will touch the third branch.*

In the *Lady's and Gentleman's Diary* for 1861, p. 70–72, two solutions of the prize question are given, and three properties, due to Stephen Watson of Haydonbridge, are appended:

(1) *Two tangents to the curve being drawn at right angles, the locus of their intersection is the circle which touches the three branches.*

(2) *Two normals being drawn at right angles, the locus of their intersection is the fixed circle.*

[The enunciation of the third property would require a diagram.]

In the *Lady's and Gentleman's Diary* for 1862, p. 65, Stephen Watson proposes the question:

*If $A$ denote the area of the locus of the intersection of two tangents to the hypocycloid of three branches, making a given angle $a$ with each other, and $A'$ that of the corresponding normals; then will $A' = 9A$ whatever be the value of $a$. And when $a = 60°$, $A'$ and $A$ become equal to the areas of the fixed and rolling circles respectively.*

Three solutions of the question are given in the *Diary* for 1863, p. 59–62.

In Crelle's *Journal*, Vol. 64, p. 101–123 (1865) there is an article: "Sur l'hypocycloïde à trois rebroussements," by Cremona.

It is dated Bologna, 10th May 1864. All Steiner's properties are there demonstrated and some others are added.

In the same *Journal*, Vol. 66, p. 344–362 (1866) there is an article by Herr Siebeck, "Ueber die Erzeugung der Curven dritter Klasse und vierter Ordnung durch Bewegung eines Punktes."

1865–1878

EDUCATIONAL TIMES REPRINT (E.T.R.)

In Vol. 3, p. 81–82 (1865) W. K. Clifford gives a solution of the question (proposed by H. R. Greer):

*To find the envelope of the straight line joining the feet of the perpendiculars drawn on the sides of a triangle from a point in the circumference of the circumscribed circle.*

In Vol. 4, p. 13–17 (1866) the Rev. R. Townsend has an article showing that nearly all the interesting results obtained by Steiner with respect to this envelope may be deduced from the two following properties of the hypocycloid of three cusps:

(1) *The chord intercepted by the inscribed circle on any tangent to the curve is trisected externally at its point of contact with the curve.*

(2) *The two arcs into which the inscribed circle is divided by any tangent to the curve are trisected internally at their points of contact with the corresponding branches of the curve.*

In Vol. 4, p. 58–59 (1866) Mr Morgan Jenkins has a short article "On the regular tricusped hypocycloid" in which he discusses its evolute, radius of curvature, length of curve and area.

In Vol. 29, p. 80–83 (1878) the question is again raised "Required the envelope of the Simson line," and three different solutions are given, the first by Mr R. F. Davis, the second by Mr W. J. C. Sharp and others, the third by Christine Ladd, Elizabeth Blackwood and others.

In a recent letter to me Mr Davis makes the remark: "I have found that a great many cases in which the tricusp occurs as an envelope depend upon this simple fact:

*If A be a fixed point on a given circle, and PQ a variable chord having a fixed direction, then the envelope of QR (drawn parallel to AP) is a tricusp concentric with and circumscribing the given circle.*"

See also Vol. 34, p. 38, and Vol. 52, p. 69.

1866–1868

QUARTERLY JOURNAL OF MATHEMATICS

In Vol. 7, p. 70–74 (1866), Henry R. Greer has an article entitled "The Geometry of the Triangle. On the equation of a certain envelope."

In Vol. 8, p. 209–211 (1867), N. M. Ferrers gives an "Investigation of the envelope of the straight line joining the feet of the perpendiculars let fall on the sides of a triangle from any position in the circumference of the circumscribed circle."

In Vol. 9, p. 31–41, 175–176 (1868), Cayley has a memoir "On a certain envelope depending on a triangle inscribed in a circle."

1866–1885

GIORNALE DI MATEMATICHE

(BATTAGLINI)

In Vol. 4, p. 214–222 (1866), G. Battaglini, "Sopra una curva di terza classe et di quarto ordine."

In Vol. 23, p. 263–284 (1885), Mr Carmelo Intrigila, "Studio geometrico sull'ipocicloide tricuspide."

1869–1902

NOUVELLES ANNALES DE MATHÉMATIQUES

In Vol. 8 (second series) p. 45 (1869) Mr H. Brocard proposes the question :

*On donne un cercle $C$ tangent à une droite $D$ en $O$. D'un point $M$ de la circonférence on mène $MA$ perpendiculaire à $OD$, et l'on prend $AB = AO$. On joint $BM$, et l'on demande l'enveloppe de la droite $BM$ quand le point $M$ se déplace sur la circonférence. L'enveloppe cherchée admet trois axes de symétrie et trois points de rebroussement remarquables.*

Solutions to the question will be found in the same volume p. 418–420, and 470–472.

In Vol. 9 (second series) p. 73–84 (1870), Paul Serret, "Sur un Théorème de M. Ferrers."

In Vol. 9 (second series), p. 202–211, 256–270 (1870), L. Painvin, "Note sur l'hypocycloïde à trois rebroussements." This note was communicated to the editors in July 1865.

In Vol. 9 (second series) p. 254–256 (1870), E. Laguerre, "Extrait d'une lettre adressée à M. Bourget."

In Vol. 9 (second series), 472 (1870), O. Callaudreau, "Théorèmes sur l'hypocycloïde à trois rebroussements."

In Vol. 14 (second series) p. 21–31 (1875), Ch. Ph. Cahen, "Sur l'hypocycloïde à trois rebroussements."

In Vol. 18 (second series) p. 33–35 (1879), Mr Badoureau, "Enveloppe de la droite de Simpson" [sic].

In Vol. 18 (second series) p. 57–67 (1879), E. Laguerre, "Sur quelques propriétés des foyers des courbes algébriques et des focales des cones algébriques."

In Vol. 6 (third series) p. 257–266 (1887), Mr Ernest Cesaro, "Sur la droite de Simson."

In Vol. 12 (third series) p. 37–65 (1893), Mr G. Humbert, "Sur l'orientation des systèmes de droites."

In Vol. 14 (third series) p. 297–304 (1895), Mr André Cazamian, "Sur les cubiques unicursales."

In Vol. 1 (fourth series) p. 168–171 (1901), E. Duporcq, "Sur l'hypocycloïde à trois rebroussements."

In Vol. 2 (fourth series) p. 206–217 (1902), M. Fréchet, "Sur quelques propriétés de l'hypocycloïde à trois rebroussements."

1870–1874

ZEITSCHRIFT FÜR
MATHEMATIK UND PHYSIK

In Vol. 15, p. 129–134 (1870), F. E. Eckhardt, "Einige Sätze über die Epicycloide und Hypocycloide."

In Vol. 17, p. 129–146 (1872), Dr L. Kiepert, "Ueber Epicycloiden, Hypocycloiden, und daraus abgeleitete Curven."

In Vol. 18, p. 363–386 (1873), Wilhelm Frahm, "Ueber die Erzeugung der Curven dritter Classe und vierter Ordnung."

In Vol. 19, p. 115–137 (1874), Milinowski, "Ueber die Steiner'sche Hypocycloide mit drei Rückkehrpunkten."

1873–1879

BULLETIN DE LA
SOCIÉTÉ MATHÉMATIQUE DE FRANCE

In Vol. 1, p. 224–226 (1873), Mr H. Brocard, "Démonstration de la proposition de Steiner relative à l'enveloppe de la droite de Simson."

In Vol. 5, p. 18–19 (1877), Mr H. Brocard, "Sur l'enveloppe de la droite de Simpson" [sic]

In Vol. 7, p. 108–123 (1879), E. Laguerre, "Sur quelques propriétés de hypocycloïde à trois points de rebroussement."

1873–1883

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

In Vol. 4, p. 321–327 (1873), Professor Wolstenholme on "Epicycloids and Hypocycloids."

In Vol. 14, p. 56–62 (1883), R. A. Roberts on "Polygons circumscribed about a Tricuspidal Quartic."

1884–1898

JOURNAL DE MATHÉMATIQUES SPÉCIALES

In this Journal, edited first by J. Bourget and then by Mr G. De Longchamps, the following articles on the three-cusped hypocycloid will be found:

1884, Vol. 3 (second series)

Mr Weill, "Note sur la droite de Simson," p. 11–16, 30–35, 57–62.

Mr G. De Longchamps, "Sur l'hypocycloïde à trois rebroussements," p. 169–178.

Mr Hadamard, p. 226–232.

1887, Vol. 1 (third series)

Mr G. De Longchamps, "Sur le Trifolium," p. 203–205, 220–223.

1889, Vol. 3 (third series)

Mr G. De Longchamps in reply to a letter from Mr E. Lemoine, p. 252–254.

1891, Vol. 5 (third series)

Mr H. Brocard, "Le Trifolium," p. 32–42, 56–64, 106–115, 123–132, 149–157, 177–181.

1892, Vol. 1 (fourth series)

Mr P. Delens, "Note sur l'hypocycloïde à trois rebroussements et sur les quartiques de troisième classe," p. 193–198.

1893, Vol. 2 (fourth series)

Mr F. Balitrand, "Aires des hypocycloïdes à trois ou à quatre rebroussements," p. 75–77.

Mr F. Balitrand, "Sur le déplacement d'une figure plane," p. 106–113.

1894, Vol, 3 (fourth series)

Mr F. Balitrand, "Quelques problèmes sur les coniques qui passent par quatre points fixes," p. 73–78, 97–101.

Mr A. Cazamian, "Théorèmes sur l'hypocycloïde à trois rebroussements," p. 78–79.

1896, Vol. 5 (fourth series)

Mr Ch. Michel, "Sur les courbes unicursales du deuxième et du troisième ordre," p. 265–269.

1897, Vol. 1 (fifth series)

Continuation of Mr Ch. Michel's article, p. 3–7, 25–28, 49–51.

Mr E. Lauvernay, "Sur la polaire de l'hypocycloïde à trois rebroussements," p. 169–177, 193–204.

Mr. Ch. Michel, "Nouveaux théorèmes sur l'hypocycloïde à trois rebroussements," p. 182–186, 204–206.

1885–1896

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY

In Vol. 3, p. 77–93 (1885), Mr John Alison, "The so-called Simson-line."

In Vol. 14, p. 122–126 (1896), Professor J. E. A. Steggall, "On the envelope of the Simson line of a polygon."

1879

S. Kantor, "Die Tangentengeometrie an der Steiner'schen Hypocycloide" in *Sitzungsberichte...der Akademie der Wissenschaften* (Vienna), Vol. 78, 2nd Division, p. 204–233 (1879).

See also by the same author, "Quelques théorèmes nouveaux sur l'hypocycloïde à trois rebroussements" in *Bulletin des Sciences Mathématiques*, Vol. 3 (second series), p. 136–144 (1879).

1882–1894

MATHESIS

In Vol. 2, p. 106–108, 122–127 (1882), Mr Barbarin "Sur la droite de Simson."

In Vol. 4 (second series), p. 62–67, 81–84 (1894), Mr F. Balitrand, "Applications d'un théorème de Chasles."

1883

Captain P. A. MacMahon, "The three-cusped hypocycloid" in the *Messenger of Mathematics*, Vol. 12, p. 151–153.

1893

Mr H. P. Nielsen, "Om de usammensatte Kurver af fjerde Orden, som daekke sig selv ved en tredie Del af en hel Omdrejning om Begyndelsespunktet" in the Danish journal *Nyt Tidsskrift for Matematik*, Section B, Vol. 4, p. 30–34.

1893–1895

MONATSHEFTE FÜR MATHEMATIK UND PHYSIK

In Vol. 4, p. 99–115 (1893), Mr G. Stiner, "Metrische Eigenschaften der Curven dritter Ordnung mit einem Doppelpunkt."

In Vol. 4, p. 135–147 (1893), Mr O. Rupp, "Ueber die mit der Parabelschaar zusammenhängenden Steiner'schen Hypocycloiden."

In Vol. 6, p. 372–374 (1895), Mr G. Stiner, "Zur Construction der Steiner'schen Hypocykloide."

1894

Some remarks on the three-cusped hypocycloid will be found in the solution given by Mr R. Gilbert to the question set at the *Concours Général de Mathématiques Spéciales* (Paris, 1894). See *Revue de Mathématiques Spéciales*, Vol. 2, p. 357–360 (1894). See also the addition to this solution by Mr A. Pagès in the same *Revue*, Vol. 3, p. 12–14 (1894). See also in the same *Revue*, Vol. 5, p. 449–451 (1900).

1895

Mr W. Godt, "Ueber den Feuerbach'schen Kreis und die Steiner'sche Curve vierter Ordnung und dritter Classe" in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 4, p. 161–162.

1896

Dr Gino Loria in his work *Il passato ed il presente delle principali teorie geometriche* (2nd edition, Turin), p. 75 (1896).

Mr W. Godt, "Ueber den Feuerbach'schen Kreis und eine Steiner'sche Curve vierter Ordnung vnd dritter Klasse" in *Sitzungsberichte der Akademie der Wissenschaften zu München*, Vol. 26, p. 119–166 (1896).

1897–1898

COMPTES RENDUS DE L'ACADÉMIE DES SCIENCES

Paul Serret has four communications, the first "Sur l'hypocycloïde de Steiner" and the others "Sur l'hypocycloïde à trois rebroussements." Vol. CXXV., p. 404–406, 423–426, 445–448, 459–461 (1897).

S. Kantor, "Réclamations de priorité à l'occasion de plusieurs Notes de M. P. Serret relatives à l'hypocycloïde à trois rebroussements." Vol. CXXVI., p. 928 (1898).

1897–1900

BULLETIN DE MATHÉMATIQUES SPÉCIALES

Mr Ch. Michel, "Théorie synthétique des cubiques à point

double et des courbes de troisième classe à tangente double." 4ᵉ année, p. 97–106 (1897–8), 5ᵉ année, p. 1–5, 65–68 (1898–9).

Mr Ch. Michel, "Sur l'hypocycloïde à trois rebroussements." 6ᵉ année, p. 103–108 (1899–1900).

1898

Dr Bücking, "Die Seitensymmetriegeraden des Dreiecks; als besonderen Fall die Steiner'sche Curve des Dreiecks" in Grunert's *Archiv der Mathematik und Physik*, Vol. 16 (second series) p. 271–319.

1901

Mr Ferdinando Paolo Ruffini, "Della ipocicloide tricuspide" in *Rendiconto della R. Accademia delle Scienze dell' Istituto di Bologna*, Vol. 5 (new series), p. 13–23.

1902

Mr A. Gob, "Note sur l'hypocycloïde à trois rebroussements" in *Mémoires de la Société Royale des Sciences de Liége*, Vol. 4 (third series), 7th memoir. Each memoir is paged separately.

1902

JOHNS HOPKINS UNIVERSITY CIRCULARS

In Vol. 22, p. 1–3, Mr H. A. Converse, "On the hypocycloids of class three inscribed in a 3-line"; and p. 4–5 "On a system of hypocycloids of class three."

1902–1904

ANNALS OF MATHEMATICS

In Vol. 3 (second series), p. 154–160 (1902), Professor R. E. Allardice, "On some curves connected with a system of similar conics."

In Vol. 5 (second series), p. 105–139 (1904), Mr H. A. Converse, "On a system of hypocycloids of class three inscribed in a given 3-line, and some curves connected with it."

In Vol. 5 (second series), p. 169–172 (1904), Professor R. E Allardice, "On a linear transformation and some systems of hypocycloids."

## Notes on Inequalities.

By V. Ramaswami, M.A.

[The following is a digest, consisting mainly of extracts, of Mr Ramaswami's paper. The author mentions that the Notes are intended for readers of Chrystal's "Algebra."]

*On a General Inequality Theorem.*

1. The important inequality of which Prof. Chrystal has given so many examples may be called the "Power Inequality."

There is a simple theorem of the Differential Calculus which is to a general function $f(x)$, what the Power Inequality is to the function $x^m$.

In what follows it will be supposed that the functions and the differential coefficients considered are finite, single-valued, and continuous between the limits of the variable considered, though they may be infinite, *at* either limit.

2. *Theorem*: If $f''(x)$ be always positive, or always negative, as $x$ increases from a value B to a value A, and if $a$ and $b$ be any two quantities lying between the limits A and B, $a$ being greater than $b$,

then
$$f'(a) \gtrless \frac{f(a)-f(b)}{a-b} \gtrless f'(b)$$

according as $f''(x) \gtrless 0$, between the limits A and B.

[Then follow several proofs of the theorem (which is practically an aspect of the Mean Value Theorem); the simplest is that obtained from consideration of the fact that the curve $y=f(x)$ is, under the specified conditions, either convex or concave to the axis of $x$ throughout the range of values considered.]

Applying the theorem to the elementary functions, we have

(i) If $x$ and $y$ be positive, and $x>y$, then

$$mx^{m-1} \gtrless \frac{x^m - y^m}{x-y} \gtrless my^{m-1}$$

according as $m(m-1) \gtrless 0$. (The Power-Inequality.)

(ii) If $a$ is any positive quantity $\neq 1$, and $x>y$,

$$a^x \log a > \frac{a^x - a^y}{x-y} > a^y \log a.$$

(iii) If $x$ and $y$ be positive, and $x>y$,

$$\frac{1}{x} < \frac{\log x - \log y}{x-y} < \frac{1}{y}.$$

(iv) If $\frac{\pi}{2} > x > y > 0$,

$$\cos x < \frac{\sin x - \sin y}{x-y} < \cos y;$$

etc.

We proceed to deduce some consequences from the general theorem.

3. *Theorem*: If $f''(x)$ be constantly positive, or constantly negative, as $x$ increases from B to A, and if $x$, $y$, $z$ be any three quantities in descending order of magnitude, lying between the limits A and B, then

$$f(x).(y-z)+f(y).(z-x).)+f(z).(x-y) \gtrless 0,$$

according as $f''(x) \gtrless 0$, between the limits A and B.

*Demonstration*: Suppose $f''(x)$ to be positive. Then, by the general theorem,

$$\frac{f(x)-f(y)}{x-y} > f'(y) > \frac{f(y)-f(z)}{y-z};$$

$$\therefore \quad \frac{f(x)-f(y)}{x-y} > \frac{f(y)-f(z)}{y-z}.$$

The denominators being positive, we have multiplying out, etc., the result

$$f(x).(y-z)+f(y).(z-x)+f(z).(x-y)>0.$$

If $f''(x)$ be negative, the inequality sign is reversed throughout.

*Examples*: (i) $f(x)=a^x$, (ii) $f(x)=x^m$, (iii) $f(x)=\log x$.

4. *Theorem*: If $f''(x)$ be constantly positive, or constantly negative, as $x$ increases from B to A, and $a$ be any fixed quantity lying between A and B, then the expression $\frac{f(x)-f(a)}{x-a}$ constantly increases, or constantly decreases, as $x$ increases from B to A (passing through the value $f'(a)$ as $x$ passes through $a$).

*Demonstration*: Suppose $f''(x)$ to be positive. Let $x$ and $y$ be any two quantities lying between the limits A and B, $x$ being greater than $y$. We have to show that

$$\frac{f(x)-f(a)}{x-a} > \frac{f(y)-f(a)}{y-a}.$$

First, if $x>y>a$, we have

$$\frac{f(x)-f(y)}{x-y} > \frac{f(y)-f(a)}{y-a};$$

Secondly, if $x>a>y$, we have

$$\frac{f(x)-f(a)}{x-a} > \frac{f(a)-f(y)}{a-y};$$

Thirdly, if $a>x>y$, we have

$$\frac{f(a)-f(x)}{a-x} > \frac{f(x)-f(y)}{x-y}.$$

And in each case the result reduces to

$$\frac{f(x)-f(a)}{x-a} > \frac{f(y)-f(a)}{y-a}.$$

If $f''(x)$ be negative, the inequality sign is reversed throughout.

*Examples*: $\quad \dfrac{x^m-a^m}{x-a}, \quad \dfrac{a^x-1}{x}, \quad \dfrac{\tan x}{x}.$

5. *Theorem*: If $f''(x)$ be constantly positive or constantly negative, as $x$ increases from B to A; and if $a, b, \dots k$ be any $n$ quantities, not all equal, lying between the limits A and B; and $p, q, \dots t$ be any system of positive multiples corresponding to $a, b, \dots k$, respectively, then

$$\frac{pf(a)+qf(b)+\dots+tf(k)}{p+q+\dots+t} \gtrless f\left(\frac{pa+qb+\dots+tk}{p+q+\dots+t}\right)$$

according as $f''(x) \gtrless 0$, between the limits A and B.

*Demonstration*: Suppose $f''(x)$ to be positive. We shall first prove the theorem in the case of two quantities $a$ and $b$. Let $a$ be $>b$. Then $x$ being any quantity between $a$ and $b$, we have

$$\frac{f(a)-f(x)}{a-x} > \frac{f(x)-f(b)}{x-b}.$$

Now, for $x$ write $\dfrac{pa+qb}{p+q}$. This is permissible as the value of this fraction lies between $a$ and $b$.

Substituting and reducing, we get

$$\frac{pf(a)+qf(b)}{p+q}>f\left(\frac{pa+qb}{p+q}\right).$$

The result is thus proved for two unequal quantities $a$ and $b$. If $a$ and $b$ be equal, the inequality becomes an equality; so that, *in any case*, we can write

$$\frac{pf(a)+qf(b)}{p+q}\nless f\left(\frac{pa+qb}{p+q}\right).$$

Hence, by induction, we obtain

$$\frac{pf(a)+qf(b)+\ldots+tf(k)}{p+q+\ldots+t}>f\left(\frac{pa+qb+\ldots+tk}{p+q+\ldots+t}\right).$$

If $f''(x)$ be negative, the inequality signs are reversed throughout.

***Examples***:

(i) $f(x)=x^m$; (ii) $f(x)=y^x$; (iii) $f(x)=\sin^x$; (iv) $f(x)=\tan x$.

[The author then points out that inequalities of a different form can be obtained by writing for $f(x)$, say, $\log f(x)$; so that constancy of sign in $f''(x)$ is replaced by that in $u\equiv f(x).f''(x)-\{f'(x)\}^2$.

The results are given for this particular case, and a great many interesting results arise out of it.

*E.g.*, 1. (i) $e^{\frac{1}{x}}<\left(\dfrac{x}{y}\right)^{\frac{1}{x-y}}<e^{\frac{1}{y}}$ if $x>y>0$.

(ii) $x^{y-z}.y^{z-x}.z^{x-y}<1$ if $x>y>z>0$.

(iii) $x^{\frac{1}{x-1}}$ constantly decreases as $x$ increases from 0 to $\infty$, passing through the value $e$ as $x$ passes through the value 1.

(iv) $(a^p.b^q\ldots k^t)^{\frac{1}{p+q+\ldots+t}}<\dfrac{pa+qb+\ldots+tk}{p+q+\ldots+t}$

where $a, b, \ldots k$ are not all equal, and the symbols all denote positive numbers.

2. From $S_x \equiv a^x + b^x + \dots + k^x$,

(iii) $\left(\dfrac{a^x + b^x + \dots + k^x}{n}\right)^{\frac{1}{x}}$ constantly increases as $x$ increases from $-\infty$ to $+\infty$, and has the limiting value $(a.b.\dots k)^{\frac{1}{n}}$ when $x = 0$.

3. From $\cos x$,

(ii) $(\cos x)^{y-z}.(\cos y)^{z-x}.(\cos z)^{x-y} < 1$, if $\dfrac{\pi}{2} > x > y > z > 0$.

(iii) $(\cos x)^{\frac{1}{x}}$ constantly decreases as $x$ increases from 0 to $\dfrac{\pi}{2}$ and has the limiting value 1, when $x = 0$.

(iv) $(\cos x)^p.(\cos y)^q. < \left(\cos\dfrac{px + qy}{p+q}\right)^{p+q}$, $\dfrac{\pi}{2} > x > y > 0$ and $p$ and $q$ positive.]

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## On Mathematical Instruments and the accuracy to be obtained with them in some elementary practical problems.

By J. H. A. M'Intyre.

*Seventh Meeting*, 9*th June* 1905.

Mr J. W. BUTTERS in the Chair.

## Note on the determination of the axes of a conic.

By R. F. DAVIS, M.A.

Let $\alpha$, $\beta$, $\gamma$ be the known trilinear coordinates (actual lengths of perpendiculars) of the centre of a conic inscribed in the triangle of reference ABC.

Since the product of the perpendiculars from the foci upon any tangent is equal to the square of the semi-minor axis ($=\rho^2$), it follows that if $x$, $y$, $z$ be the coordinates of one focus then $\rho^2/x$, $\rho^2/y$, $\rho^2/z$ are the coordinates of the other focus.

Also, since the centre bisects the join of the foci,

$$x + \rho^2/x = 2\alpha,$$
$$y + \rho^2/y = 2\beta,$$
$$z + \rho^2/z = 2\gamma.$$

Therefore
$$x = \alpha + \sqrt{\alpha^2 - \rho^2},$$
$$y = \beta + \sqrt{\beta^2 - \rho^2},$$
$$z = \gamma + \sqrt{\gamma^2 - \rho^2},$$

But
$$ax + by + cz = 2\Delta = a\alpha + b\beta + c\gamma\ ;$$

hence
$$a\sqrt{\alpha^2 - \rho^2} + b\sqrt{\beta^2 - \rho^2} + c\sqrt{\gamma^2 - \rho^2} = 0.$$

Thus, given the centre of an inscribed conic the semi-axes are determined by the above equation.

It may be noted that if $\alpha = \beta = \gamma = r$ (radius of inscribed circle), the equation becomes $(a + b + c)\sqrt{r^2 - \rho^2} = 0$, and there is only one value of $\rho^2$.

The equation cleared of radicals is

$$L\rho^4 - 2M\rho^2 + N = 0,$$

where
$$\begin{aligned} L &= 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) \\ &= 16\Delta^2, \\ M &= a^2\alpha^2(b^2+c^2-a^2) + b^2\beta^2(c^2+a^2-b^2) + c^2\gamma^2(a^2+b^2-c^2) \\ &= 2Rabc[\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C], \\ N &= (a\alpha + b\beta + c\gamma)(b\beta + c\gamma - a\alpha)(c\gamma + a\alpha - b\beta)(a\alpha + b\beta - c\gamma). \end{aligned}$$

It is to be noticed that

$$\begin{aligned} \rho_1^2 + \rho_2^2 &= (\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C)/2\sin A\sin B\sin C \\ &= \text{the square of the tangent from } \alpha, \beta, \gamma \text{ to the polar circle of ABC.} \end{aligned}$$

Thus the director circle of any conic inscribed in the triangle ABC cuts orthogonally the polar circle of the same triangle. A particular case of this is that the centre of a rectangular hyperbola inscribed in the triangle ABC lies on the polar circle.

The director circle of a variable conic inscribed in a given quadrilateral cuts orthogonally the polar circles of the four triangles formed by three out of four sides (themselves coaxal), and therefore belongs to the conjugate coaxal system (Gaskin's Theorem).

It will be seen, for example, that the axes of Steiner's ellipse can be easily found from the above equation.

Putting $a\alpha = b\beta = c\gamma = 2\Delta/3$, we get

$$\rho_1^2 + \rho_2^2 = (a^2 + b^2 + c^2)/18 \quad \text{and} \quad \rho_1^2\rho_2^2 = \Delta^2/27.$$

# A direct method of obtaining the Foci and Directrices from the general equation

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2 = 0.$$

By D. K. PICKEN, M.A.

The general equation of the second degree in two variables

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad - \quad - \quad (1)$$

can be brought by a direct process into the form

$$(x-\xi)^2 + (y-\eta)^2 = (lx + my + n)^2,$$

the determination of the constants $\xi$, $\eta$, $l$, $m$, $n$ depending only on the solution of quadratic equations; so that the method is suitable for determining the foci, directrices, and eccentricities of conics with given numerical equations.

The equation (1) may be written

$$(\lambda - a)x^2 - 2hxy + (\lambda - b)y^2 = \lambda(x^2 + y^2) + 2gx + 2fy + c$$

and if $\lambda$ is a root of the quadratic equation

$$(\lambda - a)(\lambda - b) = h^2 \text{ or } \phi(\lambda) \equiv \lambda^2 - (a+b)\lambda + ab - h^2 = 0, \quad (2)$$

$$[\text{Discriminant } \{(a-b)^2 + 4h^2\}]$$

the equation (1) becomes

$$(lx + my)^2 = \lambda(x^2 + y^2) + 2gx + 2fy + c$$

where $\qquad l^2 = \lambda - a, \quad m^2 = \lambda - b \quad \text{and} \quad lm = -h,$

or $\qquad (lx + my + \nu)^2 = \lambda(x^2 + y^2) + 2(l\nu + g)x + 2(m\nu + f)y + \nu^2 + c$

$$= \lambda\left\{\left(x + \frac{l\nu + g}{\lambda}\right)^2 + \left(y + \frac{m\nu + f}{\lambda}\right)^2\right\} \quad - \quad (3)$$

if $\nu$ be so chosen that

$$(l\nu + g)^2 + (m\nu + f)^2 = \lambda(\nu^2 + c),$$

*i.e.*, if $\nu$ be a root of the quadratic equation

$$(l^2 + m^2 - \lambda)\nu^2 + 2(gl + fm)\nu + g^2 + f^2 - \lambda c = 0 \quad - \quad (4)$$

or $$\frac{ab-h^2}{\lambda}\nu^2 - 2(gl+fm)\nu + \lambda c - g^2 - f^2 = 0$$

$\left(\text{since} \qquad \lambda - l^2 - m^2 = a + b - \lambda = \dfrac{ab-h^2}{\lambda} \text{ by } (2)\right)$,

of which the discriminant is

$$\left[g^2(\lambda-a)+f^2(\lambda-b)-2fgh-c(ab-h^2)+(g^2+f^2)\frac{ab-h^2}{\lambda}\right]$$
$$\equiv\left[(g^2+f^2)\left(\lambda+\frac{ab-h^2}{\lambda}\right)-ag^2-bf^2+ch^2-2fgh-abc\right]$$
$$\equiv(g^2+f^2)(a+b)-ag^2-bf^2+ch^2-2fgh-abc, \text{ by } (2)$$
$$\equiv -\Delta.$$

If we suppose $a$, $b$, $c$, $f$, $g$, $h$ all real and $a$ positive, the equation (1) is satisfied by *real* point-pairs unless $(ab-h^2)$ and $\Delta$ are both positive, *i.e.*, the equation (1) corresponds to a curve which can be drawn on the $xy$ plane unless $(ab-h^2)$ and $\Delta$ are both positive.

The roots of the quadratic (2) are then real and they are separated by $a$ or by $b$ (*i.e.*, neither of them lies between $a$ and $b$); hence, if their values are $\lambda_1$ and $\lambda_2$ we shall have $\lambda_1 - a$, $\lambda_1 - b$ both positive and $\lambda_2 - a$, $\lambda_2 - b$ both negative, and therefore $l_1$, $m_1$ are real numbers and $l_2$, $m_2$ are imaginary numbers.

Also $\quad l_1^2 l_2^2 = (a-\lambda_1)(a-\lambda_2) = \phi(a) = -h^2 = (b-\lambda_1)(b-\lambda_2) = m_1^2 m_2^2$

and $\therefore$ since $\qquad l_1 m_1 l_2 m_2 = h^2,$

we have $\qquad l_1 l_2 + m_1 m_2 = 0,$

*i.e.*, the two straight lines given by the equations

$$l_1 x + m_1 y = 0, \quad l_2 \iota x + m_2 \iota y = 0$$

(in which the coefficients are real numbers) intersect at right angles.

Consider separately the cases in which the equation (1) represents

(i) an ellipse, (ii) an hyperbola.

(i) *For the Ellipse*: $ab > h^2$ and $\Delta < 0$;

therefore $b$ is of the same sign as $a$, that is positive, and $(a+b)$ is positive; hence $\lambda_1$ and $\lambda_2$ are both positive.

Let $\nu_1$, $\nu_1'$ be the roots of equation (4) when $\lambda = \lambda_1$

and let $\nu_2$, $\nu_2'$ ,, ,, ,, ,, ,, ,, ,, $\lambda = \lambda$ ;

then $\nu_1$, $\nu_1'$ are real since $\sqrt{-\Delta}$ and $(gl_1 + fm_1)$ are real

and $\nu_2$, $\nu_2'$ are complex since $(gl_2 + fm_2)$ is imaginary.

Hence the two equations

$$l_1x + m_1y + \nu_1 = 0, \quad l_1x + m_1y + \nu_1' = 0$$

give "real directrices"; the corresponding "real foci" being

$$\left(-\frac{l_1\nu_1 + g}{\lambda_1},\ -\frac{m_1\nu_1 + f}{\lambda_1}\right) \text{ and } \left(-\frac{l_1\nu_1' + g}{\lambda_1},\ -\frac{m_1\nu_1' + f}{\lambda_1}\right);$$

these foci clearly both lie on the line given by

$$l_2(\lambda_1 x + g) + m_2(\lambda_1 y + f) = 0, \text{ i.e., the major axis};$$

and, by a similar process, the "imaginary foci" lie on the perpendicular line given by

$$l_1(\lambda_2 x + g) + m_1(\lambda_2 y + f) = 0, \text{ i.e., the minor axis.}$$

The "eccentricity" $e_1$ corresponding to the real foci and directrices is given by

$$e_1^2 = \frac{l_1^2 + m_1^2}{\lambda_1} = \frac{+\sqrt{(l_1^2 - m_1^2)^2 + 4l_1^2m_1^2}}{\lambda_1} = \frac{+\sqrt{(a-b)^2 + 4h^2}}{\lambda_1}.$$

(ii) *For the Hyperbola*: $ab < h^2$ and therefore $\lambda_1$ is positive and $\lambda_2$ negative; $\Delta$ may be either negative or positive:

(A) *If $\Delta$ is negative*, the work is the same as for the case (i);

$\nu_1$, $\nu_1'$ are real and the corresponding eccentricity, foci and directrices are real, while $\nu_2$, $\nu_2'$ are complex and the corresponding eccentricity, foci and directrices are not real.

(B) *If $\Delta$ is positive*,

$\nu_1$, $\nu_1'$ are complex and the corresponding foci and directrices are not real; the eccentricity is real;

$\nu_2$, $\nu_2'$ are pure imaginary numbers, therefore the equations

$$l_2x + m_2y + \nu_2 = 0, \quad l_2x + m_2y + \nu_2' = 0$$

represent straight lines, the real directrices; and the corresponding foci are

$$\left(-\frac{l_2\nu_2 + g}{\lambda_2},\ -\frac{m_2\nu_2 + f}{\lambda_2}\right) \text{ and } \left(-\frac{l_2\nu_2' + g}{\lambda_2},\ -\frac{m_2\nu_2' + f}{\lambda_2}\right)$$

and the eccentricity is given by $e^2 = \dfrac{-\sqrt{(a-b)^2 + 4h^2}}{\lambda_2}$.

In this case (B), the introduction of imaginary numbers into the determination of the real foci and directrices may be avoided by writing the original equation

$$(a+\mu)x^2 + 2h + y + (b+\mu)y^2 = \mu(x^2+y^2) - 2gx - 2fy - c$$

and proceeding as above.

*Example of Numerical Case.*

*C. Smith*, p. 210.

$$x^2 - 6xy + y^2 - 2x - 2y + 5 = 0 \qquad (\Delta \textit{ negative})$$

can be written

$$(\lambda - 1)x^2 + 6xy + (\lambda - 1)y^2 = \lambda(x^2+y^2) - 2x - 2y + 5.$$

Choose $\lambda$ to satisfy

$$(\lambda - 1)^2 = 9, \text{ so that } \lambda_1 = 4, \ \lambda_2 = -2.$$

then $\lambda_1$ gives $\quad 3(x+y)^2 = 4(x^2+y^2) - 2x - 2y + 5$

$$\text{i.e.,} \quad 3(x+y+\nu)^2 = 4\left\{\left(x + \frac{3\nu - 1}{4}\right)^2 + \left(y + \frac{3\nu - 1}{4}\right)^2\right\}$$

if $\nu$ be so chosen that $\quad (3\nu - 1)^2 = 2(3\nu^2 + 5)$

$$\text{i.e.,} \quad 3\nu^2 - 6\nu - 9 = 0 \quad \text{or} \quad \nu^2 - 2\nu - 3 = 0$$

$$\therefore \quad \nu_1 = 3, \ \nu_1' = -1.$$

The directrices are $x + y + 3 = 0$, $x + y = 1$;

the corresponding foci are $(-2, -2)$ and $(1, 1)$

and the eccentricity is $\sqrt{\dfrac{3}{2}}$.

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## The Ratio of Incommensurables in Elementary Geometry.

By Professor A. Brown.

# Edinburgh Mathematical Society.

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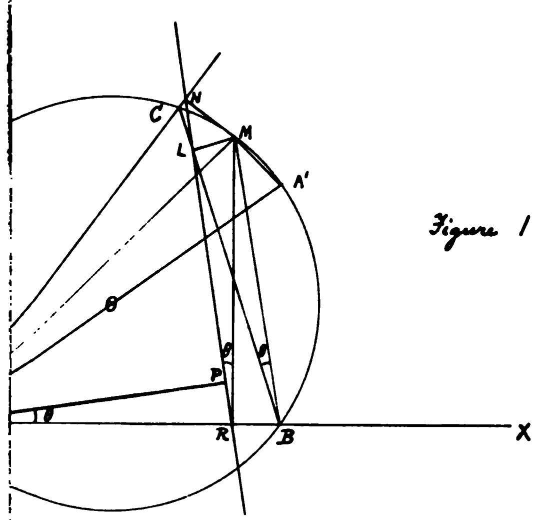


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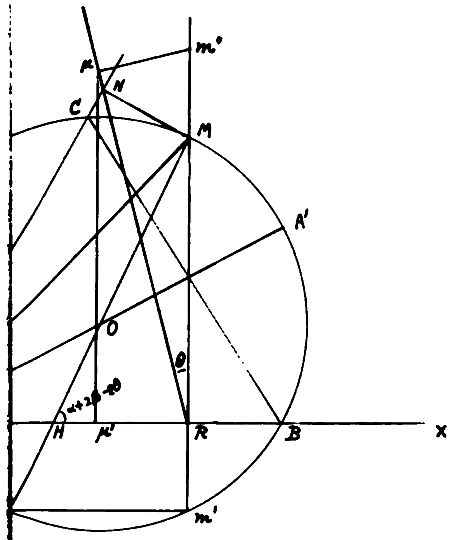


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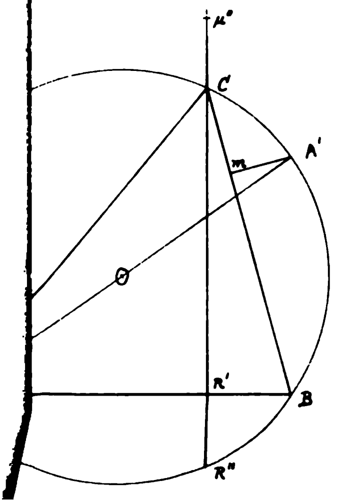


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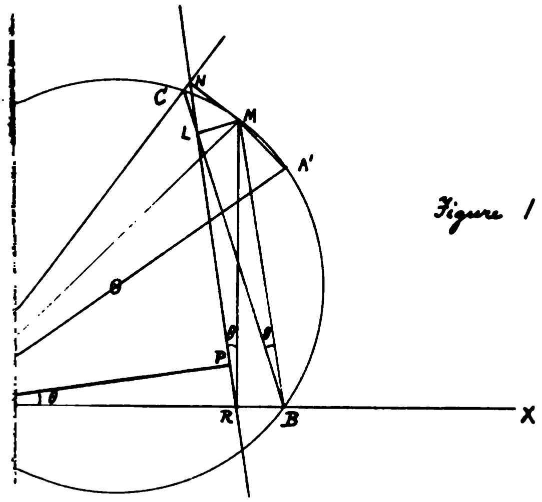


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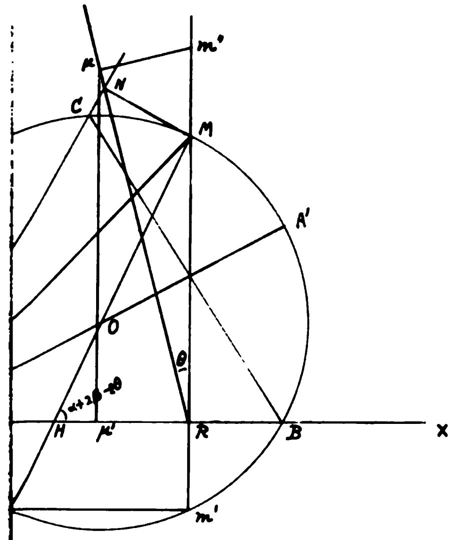


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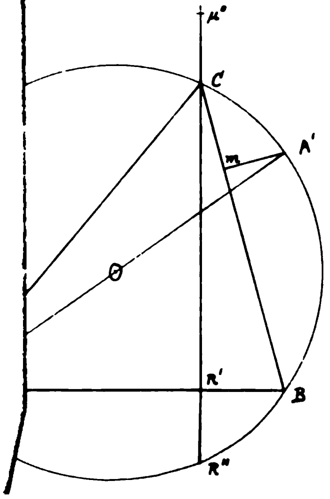


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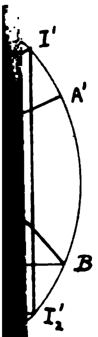


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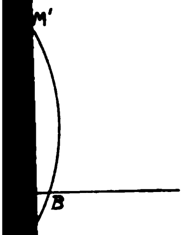


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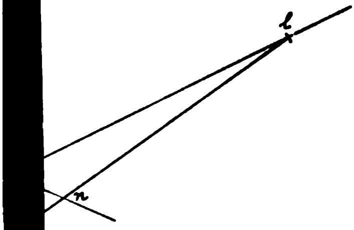


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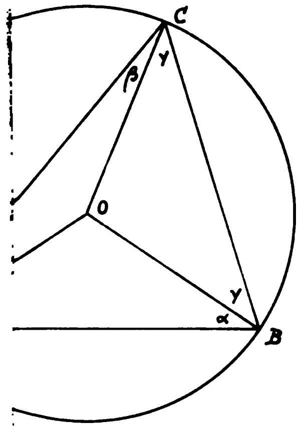


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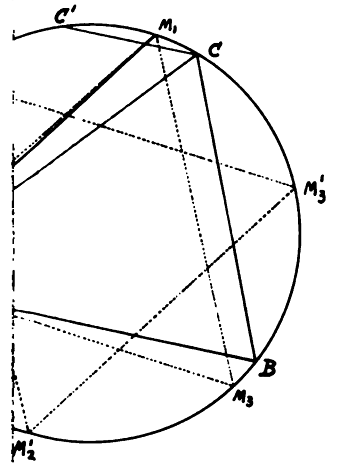


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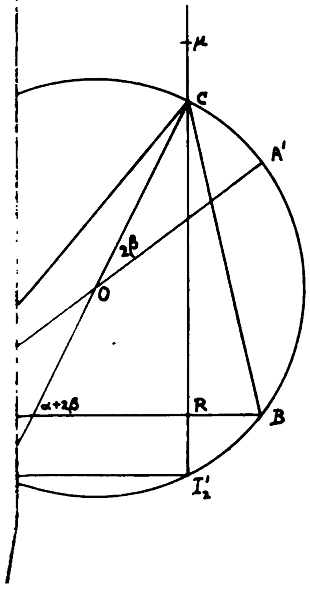


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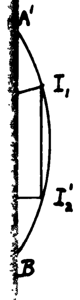


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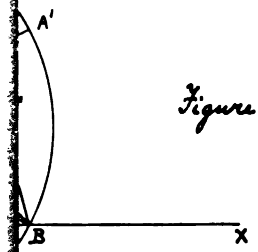
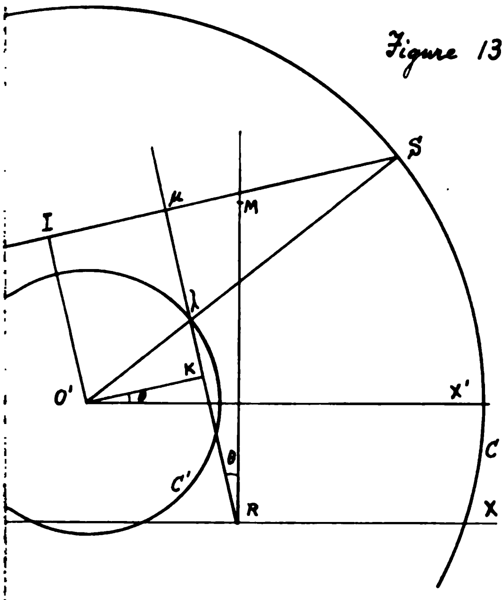



*Figure 11*


*Figure 12*

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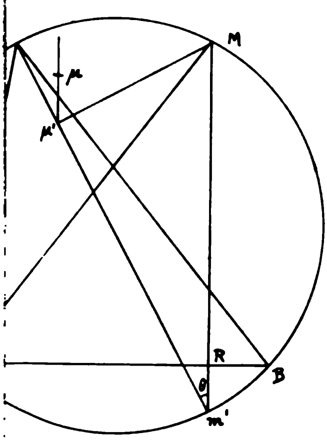


Figure 14

Figure 15

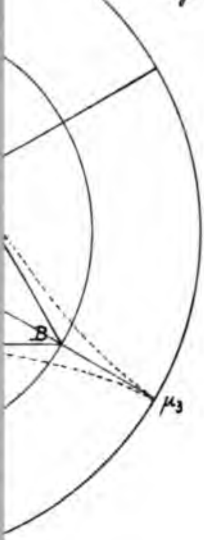


Figure 16

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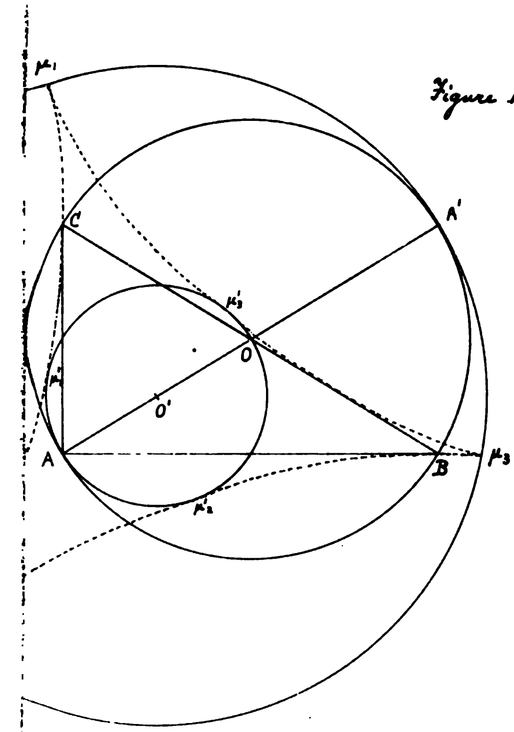


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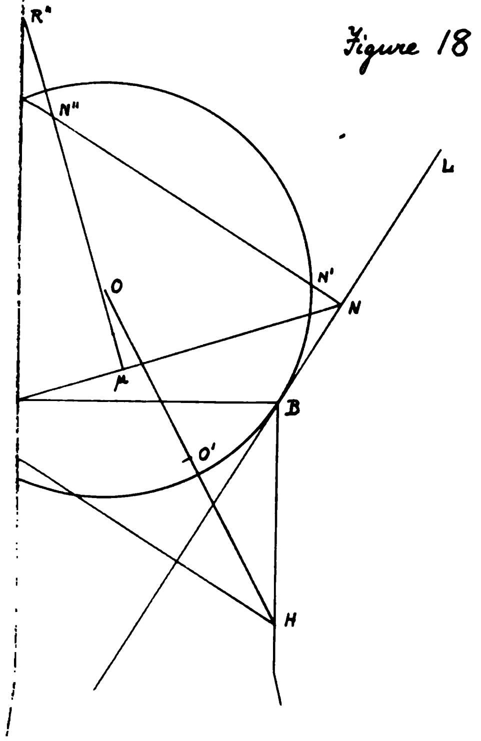


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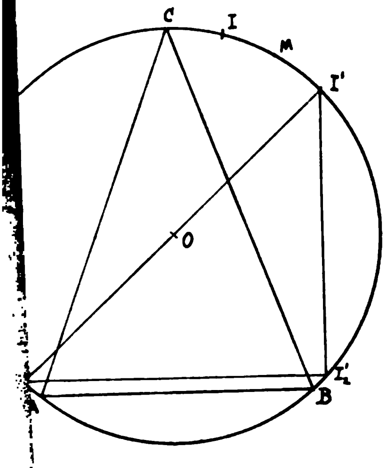
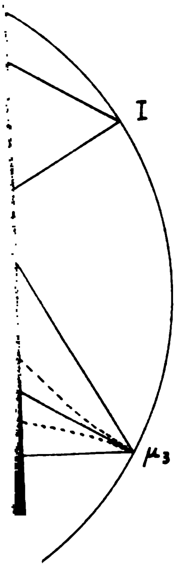
Figure 19

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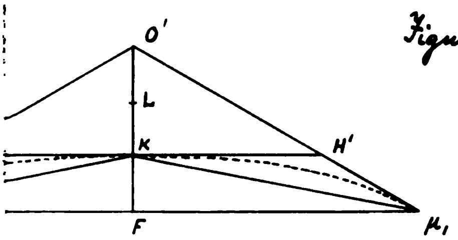


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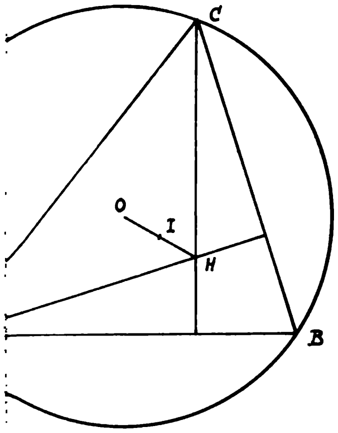


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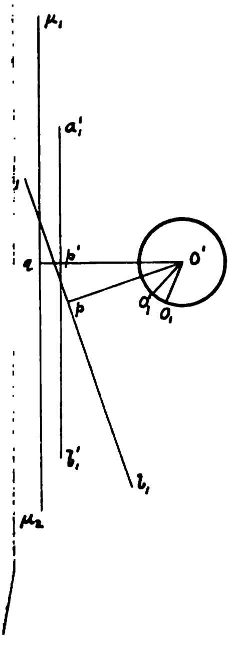


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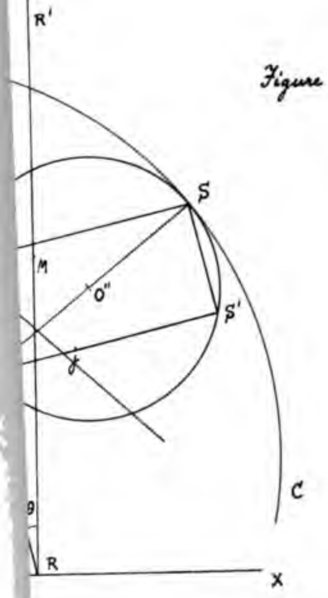


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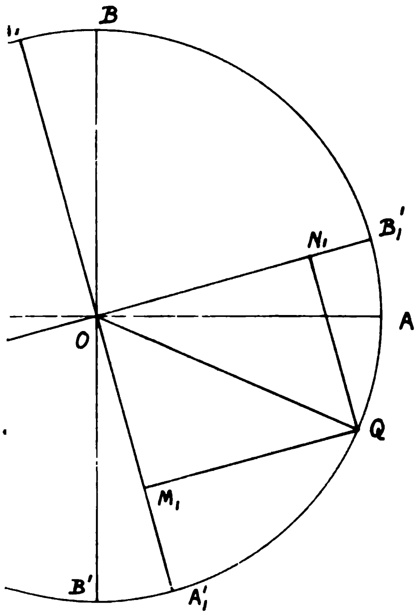
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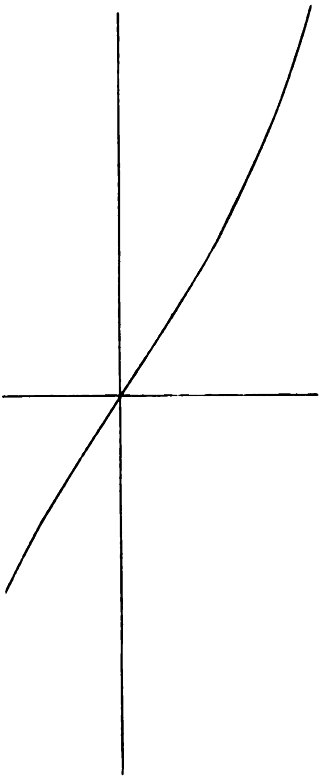

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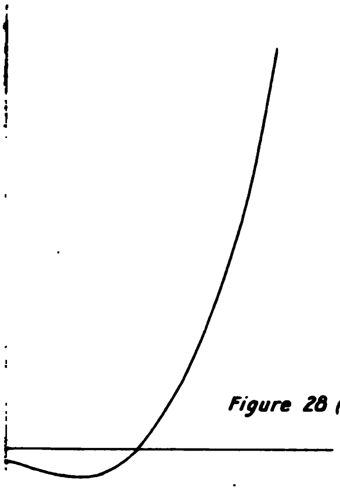

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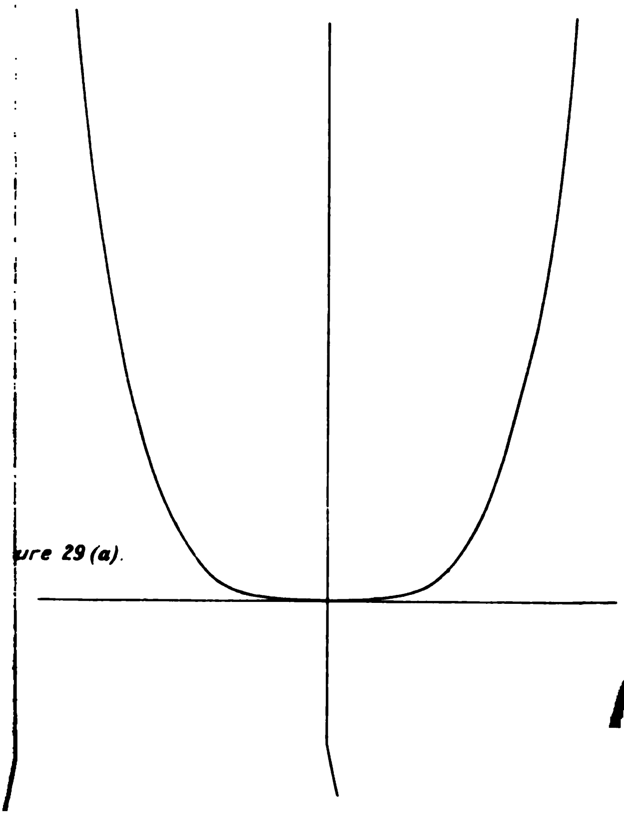

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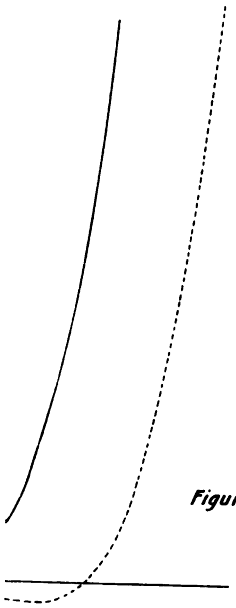
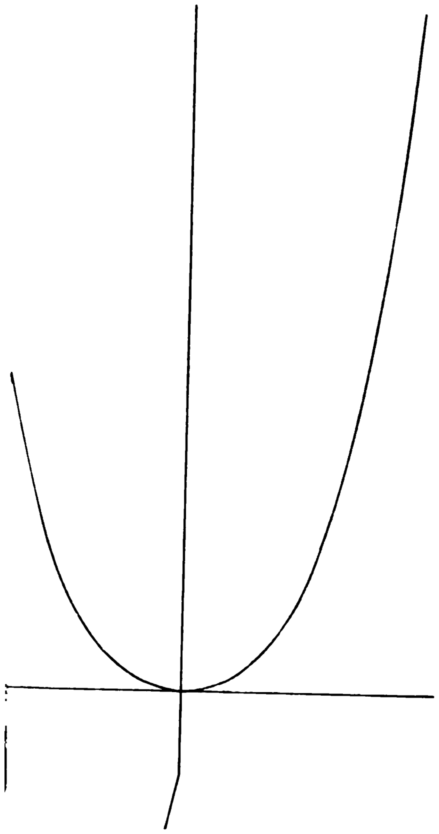


*Figure 29 (β).*

β).



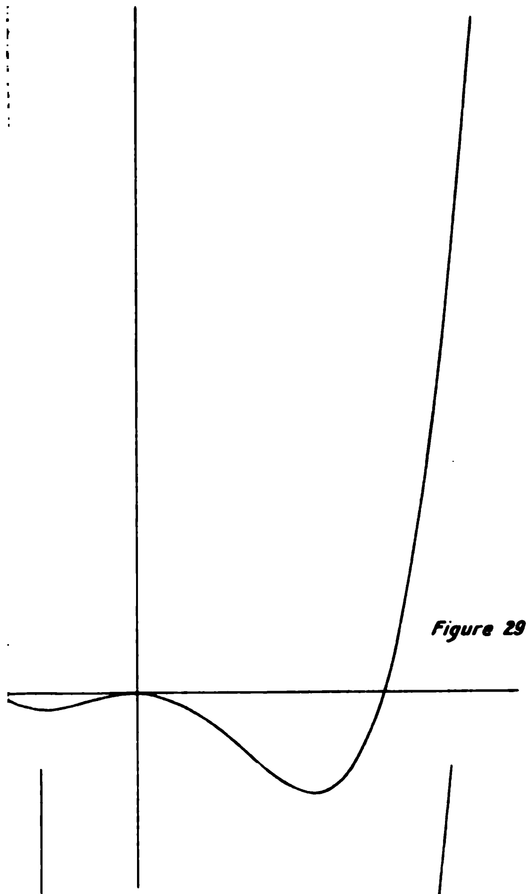


*Figure 29 (8).*

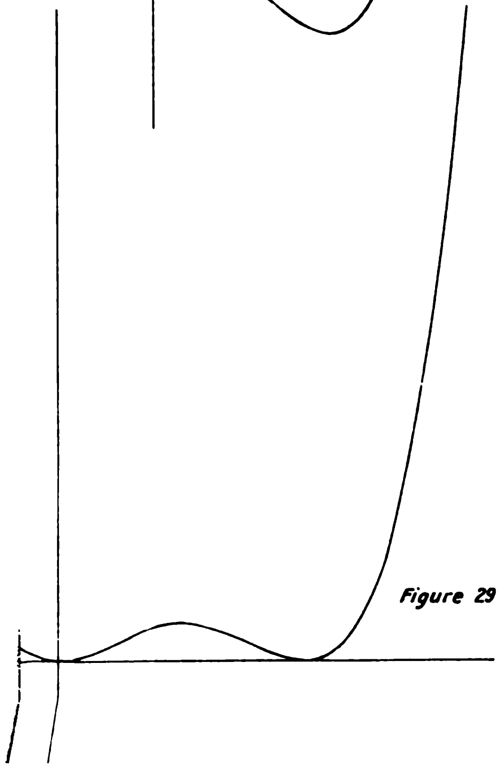
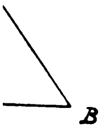


*Figure 29 (8').*


gure 30

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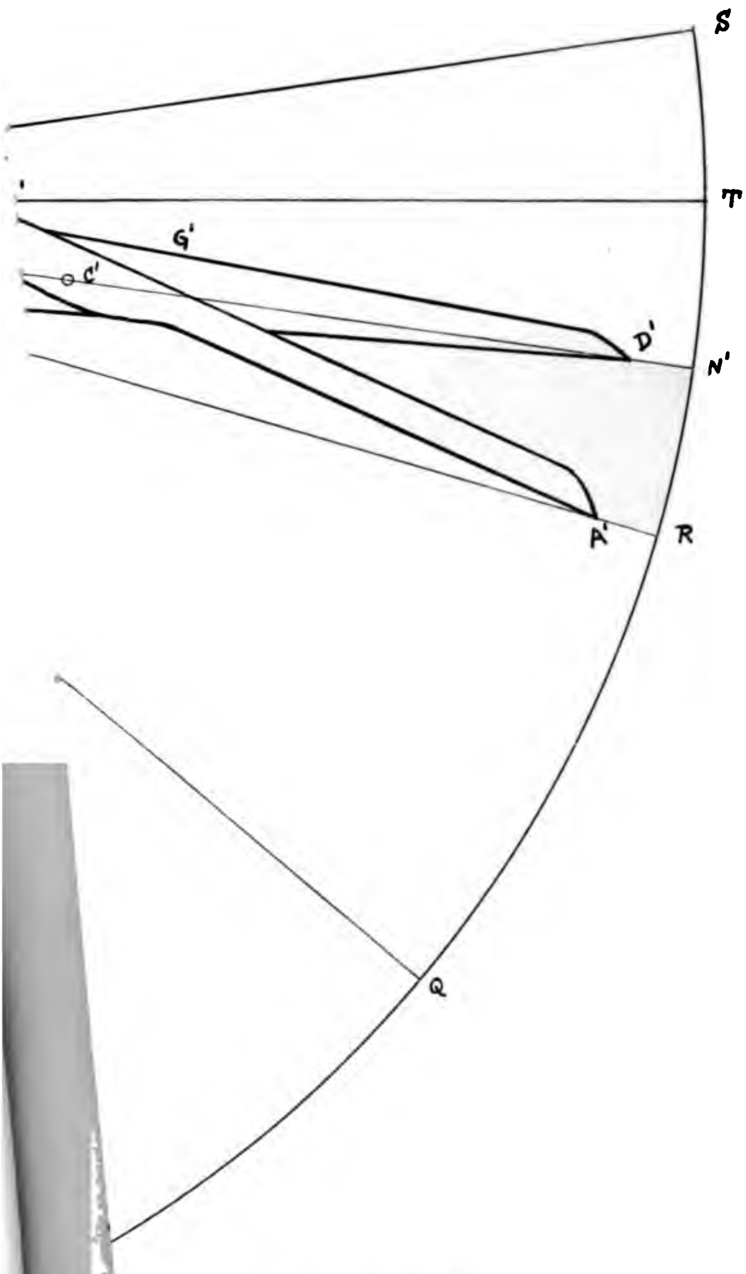

Figure 32

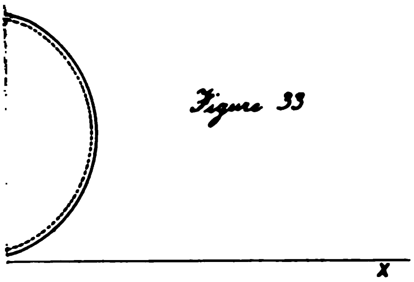


Figure 33

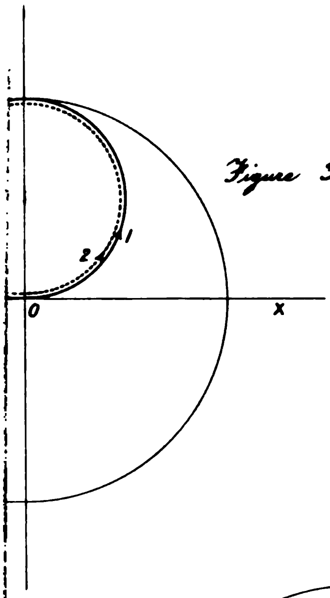


Figure 34

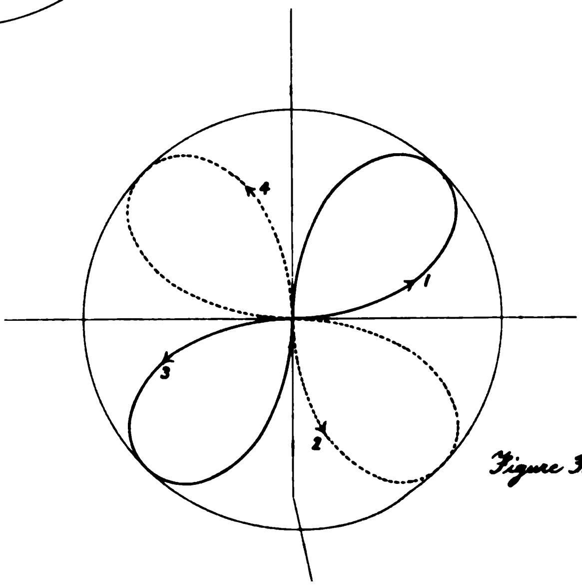


Figure 35

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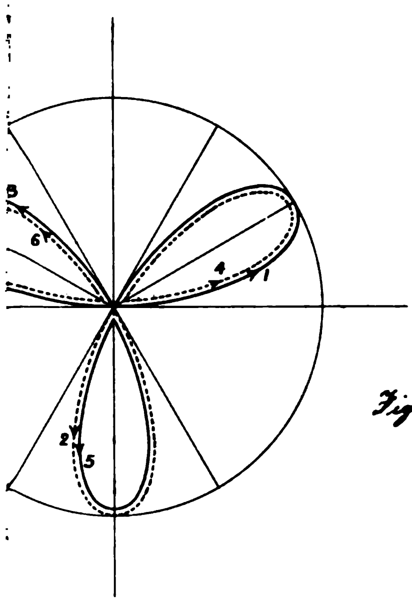


*Figure 36*

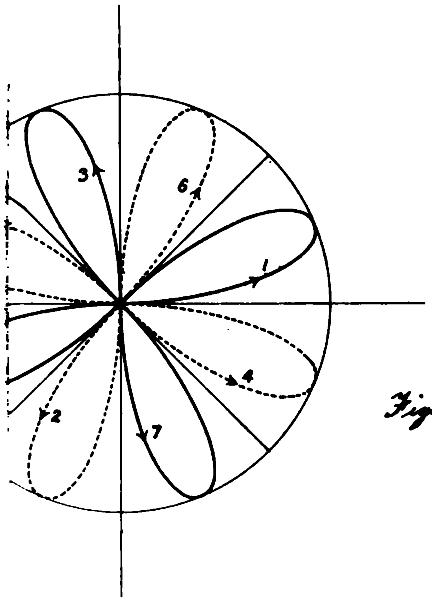


*Figure 37*

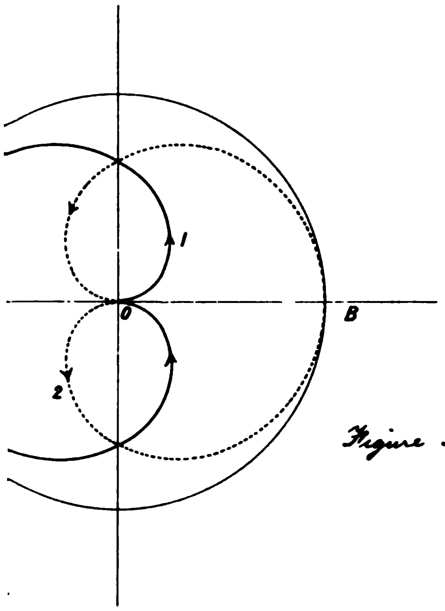


Figure 38

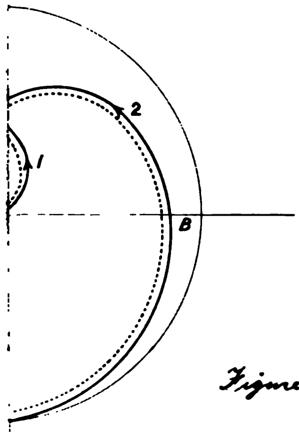


Figure 39

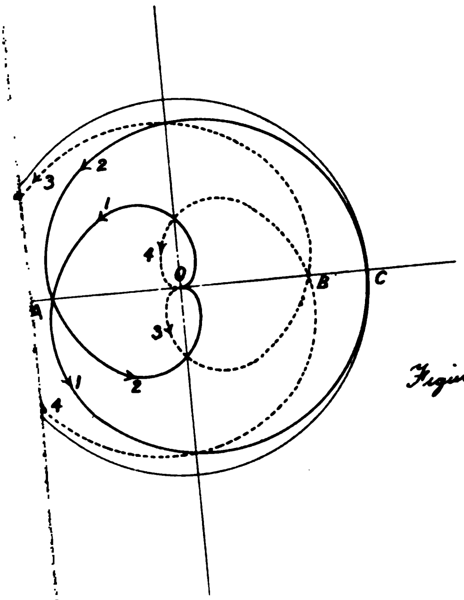


Figure 40.

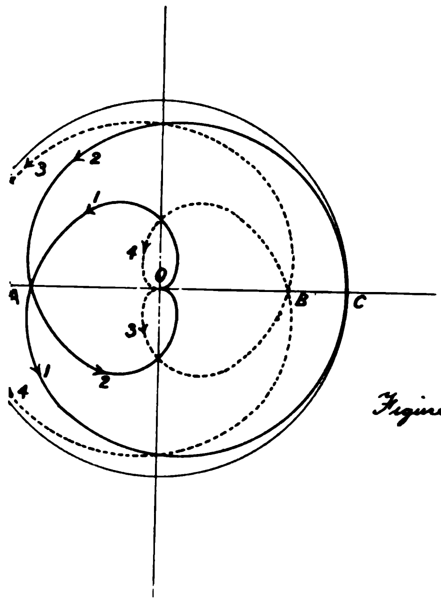


Figure 40.

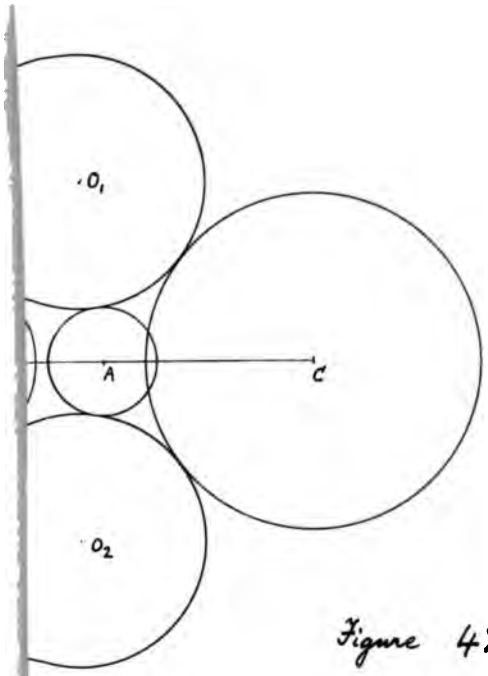


*Figure 42*

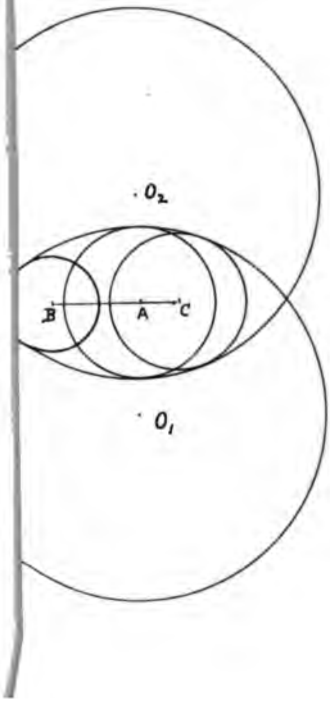


*Figure 43*

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# PROCEEDINGS

OF THE

# EDINBURGH
# THEMATICAL SOCIETY.

VOLUME XXIV.

SESSION 1905-1906.

WILLIAMS AND NORGATE,
HENRIETTA STREET, COVENT GARDEN, LONDON.
1906.

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PRINTED BY

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# INDEX.

# PROCEEDINGS

OF THE

# EDINBURGH MATHEMATICAL SOCIETY.

TWENTY-FOURTH SESSION, 1905-1906.

*First Meeting, 10th November* 1905.

W. L. THOMSON, Esq., M.A., President, in the Chair.

For this Session the following Office-bearers were elected :—

*President* - - - D. C. M'INTOSH, M.A., F.R.S.E.
*Vice-President* - - JAMES ARCHIBALD, M.A.
*Hon. Secretary* - P. PINKERTON, M.A.
*Hon. Treasurer* - JOHN TURNER, M.A., B.Sc.

*Editors of Proceedings.*

W. A. LINDSAY, M.A., B.Sc. (*Convener*).
D. K. PICKEN, M.A.

*Committee.*

W. L. THOMSON, M.A.
A. G. BURGESS, M.A., F.R.S.E.
ARCH. MILNE, M.A., B.Sc., F.R.S.E.
J. H. MACLAGAN-WEDDERBURN, M.A.
A. D. RUSSELL, B.Sc.

# On the number of Independent Conditions involved in the vanishing of a Rectangular Array.

By D. M. Y. SOMMERVILLE, M.A., D.Sc.

1. The notation for a rectangular array can be extended so as to admit of arrays in which the number of rows exceeds the number of columns.

Let
$$\left\| \begin{matrix} a_{11}, \ldots\ldots\ldots\ldots\ldots\ldots, a_{1p} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{q1}, \ldots\ldots\ldots\ldots\ldots\ldots, a_{qp} \end{matrix} \right\|_m$$
denote the aggregate of all determinants of the $m$th order which can be formed from the rectangular array of $pq$ elements by deleting $p-m$ columns and $q-m$ rows.

We may also use the abbreviated notation
$$\| a_{qp} \|_m.$$

Further, let the equation
$$\| a_{qp} \|_m = 0 \qquad \text{- - - - - - (1)}$$
denote the aggregate of equations obtained by equating each of the determinants to zero. Equations of this form are of common occurrence in the analytical geometry of $n$ dimensions, and we shall give examples from this field. (1) contains ${}_pC_m \cdot {}_qC_m$ separate equations, but not all of them are independent.

2. *The number of conditions involved in the equation* $\| a_{qp} \|_m = 0$ *is* $(p-m+1)(q-m+1)$.

Consider first the array
$$\| a_{mp} \|_m.$$

Form all the determinants which have the first $m-1$ columns the same. The number of these is $p-m+1$. Let the second subscript of the $m$th column be $\mu$. Expand each of the determinants in terms of the co-factors of $a_{\nu\mu}$. The co-factor of $a_{\nu\mu_1}$ is the same as the co-factor of $a_{\nu\mu_2} = A_\nu$ say. Equating each of these determinants to zero we get $p-m+1$ equations
$$\sum_{\nu=1}^{m} a_{\nu\mu} A_\nu = 0, \quad (\mu = m, m+1, \ldots\ldots, p).$$

Now take any other determinant of the array,

$$\begin{vmatrix} a_{1\mu_1}, & a_{1\mu_2}, & \dots, & a_{1\mu_m} \\ \dots & \dots & \dots & \dots \\ a_{m\mu_1}, & a_{m\mu_2}, & \dots, & a_{m\mu_m} \end{vmatrix}$$

Multiply the rows respectively by $A_1, A_2, \dots, A_m$ and add each to the first row. The elements of the first row will then be

$$\sum_{\nu=1}^{m} a_{\nu\mu_\lambda} A_\nu, \quad (\lambda = 1, 2, \dots, m).$$

If $\mu_\lambda = 1, 2, \dots,$ or $m-1$, this is a determinant which vanishes identically since it has two columns the same, and if $\mu_\lambda = m, m+1, \dots,$ or $p$ it is a determinant which has already been equated to zero. Hence each element of the first row vanishes and the whole determinant vanishes. The vanishing of $p-m+1$ determinants of the array is therefore sufficient, and it is also necessary, for the vanishing of all the determinants, provided no relations exist between the elements.

Hence the number of conditions involved in the equation

$$\| a_{mp} \|_m = 0$$

is $p-m+1$.*

Now we can write down all the ${}_pC_m \cdot {}_qC_m$ determinants in the form of an array, such that all those in any row or column are obtained from the same rows and columns respectively of the original array, thus

$$\begin{matrix} (1, 1) & (1, 2) \dots\dots\dots & (1, {}_pC_m) \\ (2, 1) & (2, 2) \dots\dots\dots & (2, {}_pC_m) \\ \dots & \dots & \dots \\ ({}_qC_m, 1) & ({}_qC_m, 2) \dots\dots & ({}_qC_m, {}_pC_m). \end{matrix}$$

* This theorem appears to be well known. It is stated by Cayley, "Chapters in the analytical geometry of ($n$) dimensions," *Camb. Math. Jour.*, iv., p. 119. (1843.) Dr R. F. Muirhead has pointed out to me that the general theorem which heads this section was given in answer to his Question 13651 by Professor E. J. Nanson in the *Educational Times*; see *Reprint*, vol. lxix., p. 52, and lxxi., p. 121-122 (1898-99). The theorem is there proved by a somewhat different method, and the question of the choice of determinants to be originally equated to zero is also considered. *E.g.* if a whole column of elements vanish, all the ${}_{p-1}C_{m-1}$ determinants containing this column vanish identically, and no amount of these equated to zero will be a sufficient condition for the vanishing of the others; but if only $p-m$ of the others are equated to zero this is a sufficient condition for all vanishing.

Then if we make $p-m+1$ determinants in any row vanish, the others in that row will vanish; so that if we take $p-m+1$ columns and make $q-m+1$ determinants in each vanish, all the others will vanish.

Hence the number of conditions involved in equation (1) is $(p-m+1)(q-m+1)$.

*Example*: The conditions that $s$ points $x_\nu^{(\mu)}(\mu=1, 2, \ldots, s)$ in space of $n$ dimensions should all lie in the same homaloid of $p$ dimensions ($s>p+1$) are expressed by

$$\left\| \begin{matrix} x_1', & \ldots\ldots\ldots\ldots\ldots\ldots, & x_n', & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(s)}, & \ldots\ldots\ldots\ldots\ldots\ldots, & x_n^{(s)}, & 1 \end{matrix} \right\|_{p+2} = 0.$$

This represents $(n-p)(s-p-1)$ independent conditions.

3. If the rectangular array is formed from a symmetrical determinant, as frequently happens, the number of conditions is in general fewer.

Let an array of $p$ columns and $q$ rows ($p>q$) be formed from the symmetrical determinant

$$\left| \begin{matrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{12}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n}, & a_{2n}, & \ldots, & a_{nn} \end{matrix} \right|$$

in such a way that the subscripts of the $q$ rows are all included in the subscripts of the $p$ columns. Such an array is

$$\left\| \begin{matrix} a_{11}, & a_{12}, & \ldots, & a_{1q}, & a_{1,q+1}, & \ldots, & a_{1p} \\ a_{12}, & a_{22}, & \ldots, & a_{2q}, & a_{2,q+1}, & \ldots, & a_{2p} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1q}, & a_{2q}, & \ldots, & a_{qq}, & a_{q,q+1}, & \ldots, & a_{qp} \end{matrix} \right\|$$

The number of different elements here is $pq-\frac{1}{2}q(q-1)$. The number of different determinants of the $m$th order is ${}_pC_m \cdot {}_qC_m - \frac{1}{2}{}_qC_m({}_qC_m-1)$, since the only determinants which occur twice are those symmetrical about the diagonal of the symmetrical part.

We have to find the number of conditions involved in the equation

$$\| a_{qp} \|_m = 0$$

where $a_{\nu\mu} = a_{\mu\nu}$.

4. The number of conditions involved in the equation

$$\| a_{mp} \|_m = 0$$

is evidently the same as if all the elements were different, *i.e.*, $p-m+1$.

5. Consider next the array

$$\| a_{qq} \|_m .$$

The number of distinct determinants of the $m$th order is $\frac{1}{2}{}_qC_m({}_qC_m+1)$. We can arrange these in the form of a symmetrical determinant of order ${}_qC_m$, such that all the determinants in any row or column are formed out of the same rows and columns respectively of the array.

Now, taking any row of this determinant, put $q-m+1$ of the determinant elements equal to zero; it follows, by 4, that the remaining determinants of the row vanish, and hence also all the determinants in the corresponding column. Next, in any other row put $q-m$ of the determinants equal to zero. This row has now $q-m+1$ elements zero, hence the remaining elements vanish, as also all the elements in the corresponding column. Continuing this process with $q-m+1$ rows we have all the determinants vanishing. Hence the number of independent conditions in the vanishing of the above array is

$$(q-m+1)+(q-m)+\ldots+2+1=\tfrac{1}{2}(q-m+1)(q-m+2).$$

6. Now consider the original array of $pq-\frac{1}{2}q(q-1)$ elements. The number of determinants of the $m$th order which can be formed out of it is ${}_pC_m \cdot {}_qC_m - \frac{1}{2}{}_qC_m({}_qC_m-1)$. These can be taken as the elements of a symmetrical array ${}_pC_m$ by ${}_qC_m$.

Then, just as in 5, we see that by making

$$(p-m+1)+(p-m)+\ldots+(p-q+1)$$

of the determinants vanish the remaining ones will also vanish. Hence *the number of independent conditions in the equation*

$$\| a_{qp} \|_m = 0$$

*where* $a_{\nu\mu}=a_{\mu\nu}$ *and* $p>q$ *is*

$$\tfrac{1}{2}(q-m+1)(2p-q-m+2)=(p-m+1)(q-m+1)-\tfrac{1}{2}(q-m+1)(q-m).$$

All these results can be conveniently summarised as follows :—

If $f(p, q)$ is the number of different elements of an array, whether symmetrical or with all its elements different, the number of determinants of the $m$th order is $f({}_pC_m, {}_qC_m)$ and the number of independent conditions in the vanishing of all these determinants is $f(p-m+1, q-m+1)$. If the elements are all different, $f(p, q) = pq$; if the array is symmetrical $f(p, q) = pq - \frac{1}{2}q(q-1)$.

7. *Examples*: Given the quadric locus in space of $n$ dimensions

$$\left(\begin{array}{l} a_{11}, \ldots, a_{1,n+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1,n+1}, \ldots, a_{n+1,n+1} \end{array}\right| \!\! \left( x_1, x_2, \ldots x_n, 1 \right)^2 = 0,$$

the conditions that it breaks up into two $(n-1)$-dimensional homaloids are

$$\| a_{n+1,n+1} \|_3 = 0,$$

*i.e.*, $\frac{1}{2}n(n-1)$ conditions.

If the homaloids are parallel, the conditions are

$$\left\| \begin{array}{l} a_{11}, \ldots\ldots, a_{1,n+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}, \ldots\ldots, a_{n,n+1} \end{array} \right\|_2 = 0,$$

*i.e.*, $\frac{1}{2}(n-1)(n+2)$ conditions.

If they are coincident, the conditions are

$$\| a_{n+1,n+1} \|_2 = 0,$$

*i.e.*, $\frac{1}{2}n(n+1)$ conditions.

The conditions that the locus is a cylinder, whose base is a quadric locus of $n-2$ dimensions, are

$$\left\| \begin{array}{l} a_{11}, \ldots\ldots, a_{1,n+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}, \ldots\ldots, a_{n,n+1} \end{array} \right\|_n = 0,$$

*i.e.*, 2 conditions, etc.

## Certain Series of Basic Bessel Coefficients.

By F. H. Jackson, M.A.

## A Trigonometric Dial: a Teaching Appliance.

By J. A. M'Bride, B.A., B.Sc.

*Second Meeting, 8th December* 1905.

D. C. M'INTOSH, Esq., M.A., President, in the Chair.

## A Problem of Lewis Carroll's, and the rational solutions of a Diophantine Cubic.

By C. TWEEDIE, M.A., B.Sc.

§ 1. In the *Life and Letters of Lewis Carroll* occurs the following extract from his Diary:—

"Dec. 19 (Sun).—Sat up last night till 4 a.m., over a tempting problem, sent me from New York, 'to find 3 equal rational-sided rt.-angled $\Delta$'s.' I found *two*, whose sides are 20, 21, 29; 12, 35, 37; but could not find *three*.' (*v.* page 343.)

The first object of this paper is to show how, starting from any given rational-sided right-angled triangle, we can certainly deduce other two of like area. A simple geometrical construction for a series (finite or infinite) of such triangles is also given.

§ 2. Diophantine problems of this kind have always had a great fascination for mathematicians, and the most famous of them, known as Fermat's Last Theorem, ($x^n + y^n = z^n$ has no rational solutions for $n > 2$) still awaits a satisfactory solution.

When rational solutions of an equation with rational coefficients are in question, geometrical methods of investigation may often be successfully employed. Consider the equation $f(x, y, 1) = 0$. Its solutions may be represented graphically in a plane space by means of a curve, and we have to determine "rational points" on this curve.

In the case of an equation of the first degree in $x$ and $y$ to every rational value of one coordinate corresponds a rational value of the other.

For a quadratic equation

$$(a, b, c, f, g, h \between xy1)^2 = 0$$

if one rational point $(\xi, \eta)$ exists, all the others may be found as follows. Take any "rational" line $y - \eta = m(x - \xi)$ through $(\xi, \eta)$

where $m$ has any rational value. Its second intersection with the conic represented by the quadratic equation is a rational point, and variation of $m$ will give all such points. Thus $(0, -1)$ is on the curve $x^2+y^2=1$, and therefore all other rational points are given by $2m/(1+m^2)$, $(1-m^2)/(1+m^2)$, where $m$ has any rational value. It follows that the sides of a rational-sided right-angled triangle are given by $2\rho m$; $\rho(1-m^2)$; $\rho(1+m^2)$. For the rational solutions of the equation $\alpha^2+\beta^2=\gamma^2$ correspond to the rational solutions of

$$(\alpha/\gamma)^2+(\beta/\gamma)^2=1.$$

§ 3. In the problem before us we have to find all rational solutions of the equation $\alpha^2+\beta^2=\gamma^2$, subject to the condition $\alpha\beta=2\mathrm{A}$ where A is a constant area; *i.e.*, to find the rational solutions of

$$x_1^2+y_1^2=1 \quad - \quad - \quad - \quad (1)$$

subject to the condition

$$\gamma^2x_1y_1 = 2\mathrm{A} \quad - \quad - \quad - \quad (2)$$

where $\gamma$ is a suitable rational quantity.

The rational solutions of (1) are given by

$$2m/(1+m^2),\ (1-m^2)/(1+m^2).$$

We have therefore to find all rational values of $m$ and $\gamma$ for which

$$2\gamma^2m(1-m^2)=2\mathrm{A}(1+m^2)^2. \quad - \quad (2)'$$

Write $x$ for $m$ and $y$ for $(1+m^2)/\gamma$, when we have to determine the rational points on the curve

$$x(1-x^2)=\mathrm{A}y^2. \quad - \quad - \quad (3)$$

The two sides of the corresponding triangle are $2x/y$ and $(1-x^2)/y$, and the hypotenuse is $(1+x^2)/y$.

§ 4. Description of the cubic curve.

Equation (3) represents a non-unicursal cubic, which, when A is positive, consists of an elliptic oval for $x$ between 0 and $+1$, and an infinite serpentine branch between $x=-1$, and $x=-\infty$, both symmetrical with respect to the $x$-axis.

Differentiation with respect to $x$ gives the equations

$$2\mathrm{A}yy' = 1-3x^2 \quad - \quad - \quad (4)$$

$$2\mathrm{A}y'^2+2\mathrm{A}yy'' = -6x \quad - \quad - \quad (5)$$

There is therefore no real inflexion when $x$ is positive (A positive). The inflexions have their abscissæ given by

$$x^2=(3\pm2\sqrt{3})/3$$

and there are, as in other cubics, only three real inflexions given by

$$x = -\sqrt{\{(3+2\sqrt{3})/3\}},$$

and by the point of inflexion at infinity. It is a harmonic cubic, *i.e.*, the four tangents from any point on it, touching the curve elsewhere, form a pencil of lines whose cross ratio is constant and equal to $-1$. This may be seen by taking in particular the tangents from the inflexion at infinity which cut the $x$-axis at $(-1, 0)$; $(0, 0)$; $(1, 0)$; $(\infty, 0)$. As in all other cubics consisting of two branches, there is an "even" branch and an "odd" branch. Every line cuts the oval in two points real, coincident or imaginary, or not at all. No real tangents can be drawn from the oval to touch the cubic elsewhere. Also, it will appear presently that the four tangents from any point on the "odd" branch are real. Two of the tangents touch the oval, and two touch the serpentine elsewhere. (*v.* Schröter, Theorie der ebenen Kurven dritter Ordnung.) It follows naturally that all "tangentials" are points on the odd branch. It is besides clear that, since the oval is elliptic and does not possess any real inflexion, two real tangents can be drawn to it from any point "outside" it. Regarding cubics in general it will only be necessary to assume that if $A_1B_1C_1$; $A_2B_2C_2$ are two chords of a cubic, and if $A_1A_2$, $B_1B_2$, $C_1C_2$ again cut the curve in $A_3$, $B_3$, $C_3$, then the latter three points are collinear. This theorem will be denoted as I. and quoted under the form

$$\begin{matrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{matrix} \qquad \text{- - - - -} \qquad \text{(I)}$$

§ 5. Rational solutions of a cubic.

The solution of the general cubic equation was first given in its general form by Cauchy, Exercices de Math. cahier 4. (*v.* also Desboves Nouv. Annales, 1886.) Two methods are given. In geometrical language they are as follows:—(i) If a rational point on the cubic is known, its tangential—the remaining point of intersection of the tangent *at* the point with the cubic—furnishes a second rational point; and (ii) If two rational points on a cubic are known, the line joining them cuts the cubic again in a rational point. Neither method is perfect, and either is liable to exception. Thus (i) breaks down for the point (0, 1) on $x^3+y^3=1$, for this point is a point of inflexion. Similarly (ii) breaks down when the

line joining a point and its tangential is taken. The method of (i) was virtually known long before for equations of the form $y^2=f_3(x)$. In Euler's *Algebra* (1784) will be found in addition another method of solving $y^2=f_3(x)$, which may be expressed geometrically as follows. If P is a rational point on the cubic, the parabola having 3-pointic contact at P of the form $y=a+bx+cx^2$, cuts the cubic again in a rational point.

The geometrical method can be extended to other cases which were most probably familiar to Lucas, but the extension is more apparent than real, and many receive their explanation from the residual theory of Sylvester for cubics.

These usually depend upon the fairly obvious theorem that if two curves of degrees $m$ and $n$ with rational coefficients cut in $mn-1$ rational points, the remaining point of intersection is also a rational point. Particular cases arise for contacts of different orders.

Thus if a conic has 5-pointic contact with a cubic at a rational point P, the remaining point of intersection is a rational point; similarly for a rational conic with 3-pointic contact at P and 2-pointic contact at P′, say, where P and P′ are rational points.

§ 6. We proceed to apply these methods to the cubic (3), viz.,

$$Ay^2=x(1-x^2).$$

There are three obvious rational points :—(0, 0); (1, 0); (−1, 0); but neither method of Cauchy's when applied to these gives any fresh solution. What is more, there can be no further rational points on the cubic if A is a square number, for it is a theorem as old as Fermat that the area A of a rational-sided right-angled triangle can not be a square (Legendre, Théorie des Nombres, Vol. II.).

Let us, however, take any particular triangle and obtain a suitable value for A. To this triangle will correspond a perfectly definite rational solution $(\xi, \eta)$ on the cubic $Ay^2=x(1-x^2)$; for if $\alpha$, $\beta$, $\gamma$ are the sides and hypotenuse, then

$$2\xi/\eta=\alpha\,; \quad (1-\xi^2)/\eta=\beta\,; \quad (1+\xi^2)/\eta=\gamma.$$

Hence $(1-\xi^2)/(1+\xi^2)$ is rational; and $\xi/(1-\xi^2)$ is rational. Hence $\xi^2$, $\xi$, $\eta$ are rational.

We may therefore start with this rational point as basis.

§ 7. *Tangential Method.* The tangent at $(\xi, \eta)$ to

$$Ay^2 = x(1 - x^2)$$

$$\text{is} \quad y - \eta = m(x - \xi) \qquad \text{- - (4)}$$

where $m = (1 - 3\xi^2)/2A\eta$. The $x$-eliminant is $x^3 + Am^2x^2 + \ldots = 0$. Hence if $(\xi_1, \eta_1)$ be the tangential of $(\xi, \eta)$ we obtain

$$2\xi + \xi_1 = -Am^2$$

$$\therefore \quad \xi_1 = -Am^2 - 2\xi = -(\xi^2 + 1)^2/4\xi(1 - \xi^2).$$

Or, if $\alpha$, $\beta$, $\gamma$ are the sides and hypotenuse of the first triangle

$$\xi_1 = -\gamma^2/2\alpha\beta = -(\alpha^2 + \beta^2)/2\alpha\beta = -(\alpha^2 + \beta^2)/4A. \qquad \text{- (5)}$$

It is unnecessary to calculate $\eta_1$, nor need any attention be paid to the sign of $\xi_1$. The sides of the new triangle are proportional to $2\xi_1$, $1 - \xi_1^2$, and are given by

$$\lambda(2\gamma^2/2\alpha\beta), \ \lambda(-1 + \gamma^4/4\alpha^2\beta^2).$$

Their product is $\alpha\beta$, hence

$$\lambda^2\gamma^2(\gamma^4 - 4\alpha^2\beta^2)/4\alpha^3\beta^3 = \alpha\beta,$$

so that $\qquad \lambda = 2\alpha^2\beta^2/(\alpha^2 - \beta^2)\gamma, \ \text{if} \ \alpha > \beta.$

The new triangle therefore has for sides

$$2\alpha\beta\gamma/(\alpha^2 - \beta^2); \ (\alpha^2 - \beta^2)/2\gamma,$$

and the hypotenuse is $(\alpha^4 + 6\alpha^2\beta^2 + \beta^4)/2\gamma(\alpha^2 - \beta^2)$. - - - - (6)

§ 8. To this analytical result corresponds a simple geometrical construction for the new triangle.

Let ABC be the original triangle. Let M be the middle point of the hypotenuse AB, and draw CD perpendicular to AB. Then $2\gamma . \text{MD} = \alpha^2 - \beta^2$. Hence one side of the new triangle is the segment MD. This is easily verified directly.

§ 9. But is a new triangle found? For in an ordinary right-angled triangle it is possible for MD to be equal to a side. Can $\alpha$ or $\beta$ equal $(\alpha^2 - \beta^2)/2\gamma$ when $\gamma^2 = \alpha^2 + \beta^2$ and all the quantities are rational? Let $\xi = \alpha/\gamma$, $\eta = \beta/\gamma$. Can $\alpha/\gamma$ or $\beta/\gamma = (\alpha^2 - \beta^2)/2\gamma^2$?

Can $\qquad 2\xi = \xi^2 - \eta^2; \ \xi^2 + \eta^2 = 1; \qquad$ - - - - (7)

or $\qquad 2\eta = \xi^2 - \eta^2; \ \xi^2 + \eta^2 = 1? \qquad$ - - - - (8)

The solutions are irrational, hence the new triangle obtained is always distinct from the first.

It naturally follows that the points of inflexion on the curve (3) can not be rational points.

If we denote the three rational quantities thus found as $\alpha'$, $\beta'$, $\gamma'$, the next triangle would have one side equal to

$$\pm 2\alpha'\beta'\gamma'/(\alpha'^2 - \beta'^2),$$

or $\pm 4\alpha\beta\gamma(\alpha^2-\beta^2)(\alpha^4+6\alpha^2\beta^2+\beta^4)/\{16\alpha^2\beta^2\gamma^4-(\alpha^2-\beta^2)^4\} = \pm \mathrm{D}$, say.
This can not equal $\alpha'$ or $\beta'$. Can it be equal to $\alpha$ or $\beta$?

The relation $\alpha = \pm \mathrm{D}$ leads to the equations

$$16\xi^2\eta^2 - (\xi^2-\eta^2)^4 = \pm 4\eta(\xi^2-\eta^2)(\xi^4+6\xi^2\eta^2+\eta^4)\,;\ \xi^2+\eta^2=1. \quad (9)$$

Eliminate $\xi$, and $\eta$ is a root of the equation

$$16\eta^8 - 32\eta^6 + 40\eta^4 - 24\eta^2 + 1 \pm 4\eta(1+2\eta^2-12\eta^4+8\eta^6) \quad = 0.$$

Write $y/2$ for $\eta$ when we deduce an equation in $y$,

$$y^8 + \ldots + 16 = 0, \quad - \quad - \quad - \quad - \quad (10)$$

where the coefficients are integers. Any rational solution of this equation must be an integer, and can therefore only be $\pm 1$; $\pm 2$; $\pm 4$; $\pm 8$; $\pm 16$; so that any rational root in $\eta$ must be $\pm 1/2$; $\pm 1$; etc. But $\xi = \pm \sqrt{1-\eta^2}$, and can be rational for only one of these values, viz., when $\eta = \pm 1$. But $\xi$ would then be zero, which is impossible from the nature of the problem. On writing $\beta = \pm \mathrm{D}$, the same equation is obtained in $\xi$ and similar conclusions are deduced. The final conclusion therefore is that the three triangles thus found are distinct equivalent and rational-sided right-angled triangles, and Carroll's problem is therefore solved. If the sides are to be integers, a suitable numerical factor can always be introduced. Owing to the restriction that the solutions must always be rational, it is very probable that the series could be indefinitely increased, but it is quite easy to construct a cubic such that even the third tangential of a point on it coincides with the point itself for certain positions on the cubic.

§ 10. The application of the chord residue method (ii) of Cauchy leads to some interesting conclusions.

It is also noteworthy that in this case the solutions obtained by the tangential method may be found by the second method.

More generally, if three points A, B, C on a cubic are known no one of which is a tangential of another, the tangential of A, say, may be found as follows :

Let AB and AC cut the curve again in B′ and C′. Then by Theorem I. we have the collinear points given by

$$\begin{array}{ccc} A & B & B' \\ A & C & C' \\ T & D & D' \end{array}$$

where D and D′ are the points in which BC and B′C′ again cut the cubic, and T is the tangential of A.

Now the cubic

$$Ay^2 = x(1-x^2)$$

possesses three rational points (0, 0); (−1, 0); (1, 0) which may be denoted by O, $O_1$, $O_2$. If therefore a rational point $P_1$ distinct from these is known, the tangential $Q_1$ of $P_1$ can be found by the residual method. It is remarkable, however, that although new rational points on the curve are found by joining $P_1$ to the points "O," no new solution of the problem is thereby directly obtained.

Let $(\xi, \eta)$ be the coordinates of $P_1$ and let $P_1O$, $P_1O_2$, $P_1O_1$ cut the curve again in $P_2$, $P_3$, $P_4$. It is easily shown that these points are

$$(-1/\xi,\ -\eta/\xi^2);\ ((\xi+1)/(\xi-1),\ 2\eta/(\xi-1)^2);\ ((1-\xi)/(1+\xi),\ 2\eta/(\xi+1)^2).$$

Consider the ratio of the sides of the triangle corresponding to $P_1$. It is given by $2\xi/(1-\xi^2)$.

Now the solutions in $x$ of the equation

$$2x/(1-x^2) = \pm 2\xi/(1-\xi^2) \text{ are } \pm\xi,\ \pm 1/\xi$$

and those of

$$2x/(1-x^2) = \pm(1-\xi^2)/2\xi \text{ are } \pm(1+\xi)/(1-\xi) \text{ and } \pm(1-\xi)/(1+\xi).$$

The ratio of the sides is therefore unaltered by selecting $P_2$, $P_3$, or $P_4$, and as the area is unaltered no new triangles are formed.

§ 11. There can likewise be no new solutions found by joining $P_2$, $P_3$, $P_4$ to the neutral points O, etc., but the number of points found in this way is limited. If $P_1'$, $P_2'$, etc., are the images of $P_1$, $P_2$, etc., in the $x$-axis, the following table contains only eight distinct points P.

| | | | |
|---|---|---|---|
| (1) | $P_1OP_2$ ; | $P_1O_1P_4$ ; | $P_1O_2P_3$ |
| (2) | $P_2OP_1$ ; | $P_2O_1P_3'$ ; | $P_2O_2P_4'$ |
| (3) | $P_3OP_4'$ ; | $P_3O_1P_2'$ ; | $P_3O_2P_1$ |
| (4) | $P_4OP_3'$ ; | $P_4O_1P_1$ ; | $P_4O_2P_2'$ |

with four similar rows formed by interchanging dashed and undashed letters P,—a transformation following from the symmetry of the cubic.

Theorem I. readily establishes these, or they may be verified analytically. Thus, to establish $P_2O_1P_3'$, we have the system

$$\begin{array}{ccc} P_2 & O & P_1 \\ O_1 & O & O_2 \\ & \infty & P_3, \end{array}$$

where $\infty$ denotes the point at infinity on the $y$-axis where the tangent at O again cuts the curve. But the line joining $P_3$ to this point is perpendicular to the $x$-axis, and therefore passes through $P_3'$. In this way groups of eight points are obtained. We proceed to examine a group of these in detail, and to apply method (ii) to them.

§ 12. If $Q_1$ is the tangential of $P_1$, it is also the tangential of $P_2'$, $P_3'$, $P_4'$; and $Q_1'$ is the tangential of $P_1'$, $P_2$, $P_3$, $P_4$.

For by I. we have the array

$$\begin{array}{ccc} P_1 & O & P_3 \\ P_1 & P_3 & O_2 \\ Q_1 & P_4' & P_4' \end{array}$$

which proves that $P_4'$ has $Q_1$ for tangential.

*Cor.* If Q is the tangential of a rational point, the four tangents that can be drawn from it to touch the curve elsewhere are rational. Or, if one of the tangents from a point on the curve is rational, so are the other three, and each meets the curve in rational points.

§ 13. The chord residue method will, in fact, be found less fruitful in new results than might have been expected.

Consider the chords

$$P_1P_2'; \quad P_1P_3'; \quad P_1P_4'.$$

Let $Q_1$ when joined to the neutral points O, $O_1$, $O_2$ give rise to the group $(Q_1 \ldots\ldots Q_4')$.

We then find the following triads

$$P_1P_2'Q_2', \quad P_1P_3'Q_3', \quad P_1P_4'Q_4'.$$

To establish the first of these we have

$$\begin{array}{ccc} P_1 & P_2' & \\ P_1 & P_3 & O_2 \\ Q_1 & O_1 & Q_4 \end{array} \qquad \therefore \; P_1 \;\; P_2' \;\; Q_2'.$$

Similarly from $P_2$ we obtain

$$P_2P_3Q_4\,;\quad P_2P_4Q_3\,;\quad P_2P_1'Q_2\,;$$

and from $P_3$ $\qquad P_3P_4Q_2$ and $P_3P_1'Q_3$.

The possible new triads for $P_1'$, etc., may be obtained by symmetry.

There results no new triangle distinct from that for $Q_1$ by joining points P.

§ 14. Let $R_1$ be the second tangential of $P_1$ and the first tangential of $Q_1$. The preceding will now apply to the group of points Q. Consider the new points to be found by joining a P and a Q.

Let $Q_1P_2$ cut the cubic again in $X_2$, and let $X_2O$ cut again in $X_1$. Form the octad of points corresponding to $X_1$.

The lines joining $Q_1$ to $P_1$, $P_2'$, $P_3'$, $P_4'$ lead to no new point, and we therefore should discuss $Q_1P_2$, $Q_1P_3$, $Q_1P_4$, $Q_1P_1'$.

It will be shown presently that these lead to $X_2X_3X_4X_1'$, *i.e.*, to a system of points possessing a common tangential.

§ 15. Use Theorem I. for the array in which the first row corresponds to a tangent from $Q_1$, the second row to the line $P_2P_1'Q_2$, and there results

$$Q_2P_1X_2'\,;\quad Q_2P_2'X_1\,;\quad Q_2P_3'X_4'\,;\quad Q_2P_4'X_3'.$$

Replace the second row by $P_2P_4Q_3$, and we find

$$Q_3P_1X_3'\,;\quad Q_3P_2'X_4'\,;\quad Q_3P_3'X_1\,;\quad Q_3P_4'X_2'.$$

Take $P_2P_3Q_4$ for the second row of I., and we find

$$Q_4P_1X_4'\,;\quad Q_4P_2'X_3'\,;\quad Q_4P_3'X_2'\,;\quad Q_4P_4'X_1.$$

In these we may interchange dashed and undashed letters.

Hence
$$\begin{array}{ccc} Q_1 & P_2' & P_2' \\ P_3 & O & P_1 \\ X_3 & P_4 & Q_2'. \end{array}$$
*i.e.*, $Q_1P_3X_3$.

We also obtain the arrays

$$\begin{array}{ccc} Q_1 & P_2' & P_2' \\ P_4 & O_1 & P_1 \\ X_4 & P_3 & Q_2' \end{array} \quad \text{and} \quad \begin{array}{ccc} O & O_1 & O_2 \\ Q_2 & P_4' & X_3' \\ Q_1 & P_1' & X_1', \end{array}$$

hence $Q_1P_4X_4$; $Q_1P_1'X_1'$.

The other joins of P's and Q's are already accounted for.

§16. The tedious process of the preceding paragraph may be somewhat curtailed by the following considerations along with a proper arrangement of the points.

The rational points so far obtained are

$$(OO_1O_2\infty);\ (P_1P_2'P_3'P_4');\ (P_1'P_2P_3P_4);\ (Q_1Q_2'Q_3'Q_4');\ (Q_1'Q_2Q_3Q_4);$$
$$(X_1X_2'X_3'X_4');\ (X_1'X_2X_3X_4);$$

in which members possessing a common tangential are grouped. It will be seen that if *any* member of one group of four points is joined to another group of four points, the same group of four points has been obtained. This follows from the following more general theorem :—

*If any member of a group of four points possessing a common tangential is joined to a similar group of four points, the same four points of intersection of joins with cubic are obtained and the latter possess a common tangential.*

Let A, B, C, D be four points on a cubic having the common tangential T. Let P be any other point, and let PA cut again in $A_1$. Let the tangential of P be Q and of A be T. Then by Theorem I.

$$\begin{matrix} P & P & Q \\ A & A & T \\ A_1 & A_1 & T_1; \end{matrix}$$

∴ $QTT_1$ are in a line.

But Q is fixed and T is fixed; therefore $T_1$ is a fixed point, and the same point $T_1$ is the tangential of $B_1$, $C_1$, $D_1$. Also the point Q is the same for the four points P possessing Q as a common tangential. Hence the theorem follows.

§17. It will be observed that the points $OO_1O_2\infty$ form such a system of four points, their tangential being the inflexional point at infinity. It may also be noted that the methods of proof hitherto employed would apply to any non-singular cubic, only for images of points the corresponding harmonic conjugates with respect to a point of inflexion require to be taken. So far as our problem is concerned no distinction is made among points possessing a common tangential.

The following notation may therefore be used with the object of finding fresh solutions.

Let P denote indifferently any one of the four points $P_1$, $P_2'$, $P_3'$, $P_4'$; and $\bar{P}$ its image (or harmonic conjugate).

Let the successive tangentials of P be Q, R, S; and ∴ of $\bar{P}$ be $\bar{Q}$, $\bar{R}$, $\bar{S}$. Let $P\bar{Q}$ cut in X, $Q\bar{R}$ in Y, PR in Z, PY in U, $R\bar{S}$ in $\zeta$.

§ 18. The following table of collinear points may then be easily constructed.

(i) $P\bar{P}O$ ; $PQP$ ; $P\bar{Q}X$ ; $PRZ$ ; $P\bar{R}\bar{X}$ ; $PYU$ ; $P\bar{Y}\bar{Z}$ ; $PS\bar{U}$.

(ii) $\overline{\bar{P}PO}$ ; $\overline{PQP}$ ; etc.

(iii) $Q\bar{Q}O$ ; $QRQ$ ; $Q\bar{R}Y$ ; $QX\bar{Z}$ ; $Q\bar{X}\bar{P}$ ; $Q\bar{Y}\bar{S}$ ; $QZ\bar{U}$.

(iv) $\overline{\bar{Q}QO}$ ; $\overline{QRQ}$ ; etc.

(v) $RRS$ ; $R\bar{R}O$ ; $RX\bar{P}$ ; $R\bar{X}U$ ; $R\bar{Y}\bar{Q}$ ; $RZP$ ; $R\bar{S}\zeta$.

(vi) $\overline{RRS}$ ; etc.

(vii) $XXY$ ; $X\bar{X}O$ ; $XZ\bar{S}$ ; $X\bar{Z}Q$.

(viii) $\overline{XXY}$ ; etc.

(ix) $YY\zeta$ ; $ZU\bar{\zeta}$ ; etc.

All the possible solutions thus obtained for Carroll's problem would not be greater than nine in number, as corresponding to

P, Q, R, S, X, Y, Z, $\zeta$, U.

§ 19. It will be observed that Y and $\zeta$ are successive tangentials of X.

For

$$\begin{array}{cccc} & P & P & Q \\ & \bar{Q} & \bar{Q} & \bar{R} \\ \therefore & X & X & Y. \end{array}$$

In the construction of the preceding table (§ 18) the following theorems are also useful.

Let P and A be any two points on the cubic, and let PAB, $P\bar{A}C$ be collinear points on the cubic. Then BC passes through a point $\bar{Q}$.

For

$$\begin{array}{cccc} & P & P & Q \\ & A & \bar{A} & O \\ \therefore & B & C & \bar{Q}. \end{array}$$

Also the residual points corresponding to AC and $\bar{A}B$ are in a line with Q.

For

$$\begin{array}{cccc} & P & A & B \\ & P & C & \bar{A} \\ \therefore & Q & \cdot & \cdot \end{array}$$

§ 20. It might readily be imagined that a convenient algorithm for finding new solutions would be found as follows.

Let $P_1AB$ be three rational points on the cubic and use the residual method to determine the points $P_2$, $P_3$... from the table

$$\begin{array}{lll} P_1 & A & P_2 \\ P_2 & B & P_3 \\ P_3 & A & P_4 \\ P_4 & A & P_5, \text{etc.} \end{array}$$

Unfortunately the very first case one takes breaks down rapidly.

Take the points $P_1$, $O_1$, $O$.

We find

$$\begin{array}{lll} P_1 & O_1 & P_4 \\ P_4 & O & P_3' \\ P_3' & O_1 & P_2 \\ P_2 & O & P_1 \end{array}$$

and we only obtain the four points $P_1P_4P_3'P_2$.

§ 21. This is a particular case of the following theorem. If A and B have a *common tangential* and we start with $P_1$ any point on the cubic, we obtain

$$\begin{array}{lll} P_1 & A & P_2 \\ P_2 & B & P_3 \\ P_3 & A & P_4 \\ P_4 & B & P_1. \end{array}$$

For let the common tangential of A and B be T, and assume the first three rows furnishing $P_2P_3P_4$ to prove $P_4BP_1$.

We find

$$\begin{array}{lll} P_1 & A & P_2 \\ P_3 & A & P_4 \\ \therefore\ R & T & S, \end{array}$$

say, where R and S are the residuals of $P_1P_3$ and $P_2P_4$ respectively.

Also

$$\begin{array}{lll} P_2 & B & P_3 \\ P_4 & B & ? \\ S & T & R. \end{array}$$

$\therefore$ $P_3R$ and $P_4B$ cut in the same point $P_1$.

This theorem is Prop. XVI. of Maclaurin's Treatise on the General Properties of Geometrical Lines, and contains the germ of what are generally termed Steiner's Polygons, viz. :—

"Let A and B be two points on a cubic, $P_1$ any point such that the system

$$\begin{array}{ccc} P_1 & A & P_2 \\ P_2 & B & P_3, \text{ etc.,} \end{array}$$

begins to repeat after $P_{2n}$, then this will happen for any other point P on the cubic." (*vide* Schröter *l. c.*)

The conditions under which this happens are furnished by the same authority.

§ 22. It might be expected that if we had three points A, B, C and a point P, then by forming the system

$$\begin{array}{ccc} P_1 & A & P_2 \\ P_2 & B & P_3, \text{ etc.,} \end{array}$$

we should obtain better results.

But if ABC are to be repeated cyclically, only five new points are obtained, and the points repeat after $P_6$. In this case A, B, C may be any three points whatsoever on the curve.

Form the table

$$\begin{array}{ccc} A & P_1 & P_2 \\ B & P_2 & P_3 \\ C & P_3 & P_4 \\ A & P_4 & P_5 \\ B & P_5 & P_6 \\ C & P_6 & ?. \end{array}$$

From these we deduce the array

$$\begin{array}{ccc} P_1 & A & P_2 \\ P_6 & P_5 & B \\ C & P_4 & P_3 \end{array} \qquad \therefore\ C \quad P_6 \quad P_1;$$

*i.e.*, we come back to the point $P_1$ from which we started.

This again is the first of a series of theorems.

"If $n$ is an odd number, $A_1, A_2, \ldots A_n$, $n$ points on a cubic, P any other point on the cubic, the polygon formed as in the preceding closes up at $P_{2n}$ after cyclical use of the points A twice." (Schröter *l. c.*)

## Solution of the Cubic Equation.

By M. Edouard Collignon.

(Abstract.)

§ 1. The roots of the cubic equation

$$x^3 + px + q = 0$$

are given by the formula

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This solution is of little practical use when the roots of the cubic are all real and unequal, that is, when $\frac{q^2}{4} + \frac{p^3}{27}$ is negative (*the Irreducible Case of Cardan's Solution*).

The object of this paper is to show how the real roots of the cubic may be found by making use of tables of values of the functions $x^3$, $x^3 + x$, $x^3 - x$.

§ 2. By substituting $\theta x'$ for $x$ in the equation

$$x^3 + px + q = 0,$$

and making $\left|\frac{p}{\theta^2}\right| = 1,$

that is, choosing $\theta = \pm\sqrt{p}$,

the given equation can be reduced to one of the three forms

$$\text{I.}\ \begin{cases} x^3 + x = \text{A}, \\ x^3 \phantom{+ x} = \text{A}, \\ x^3 - x = \text{A}; \text{ where A is positive.} \end{cases}$$

The abscissæ of the points in which the straight line $y = \text{A}$ cuts the three curves

$$y = x^3 + x,$$
$$y = x^3 \quad ,$$
$$y = x^3 - x,$$

are the real roots of the equations I.

FIGURE 1.

If $A < MP$, P being a turning-point of the curve $y = x^3 - x$, the equation

$$x^3 - x = A$$

has three real roots;

if $A = MP$, the equation has three real roots, two of which are equal;

if $A > MP$, the equation has only one real root.

§ 3. Discussion of the curves

$$y = x^3 - x, \qquad (1)$$
$$y = x^3 \quad , \qquad (2)$$
$$y = x^3 + x. \qquad (3)$$

(1) The ordinate for the curve $y = x^3$ is the mean of the ordinates for the other two curves; the same is true of $\frac{dy}{dx}$ and of $\int y\,dx$.

(2) Hence the following theorem :—

The tangents drawn to the three curves at points which have the same abscissa are concurrent.

FIGURE 2.

If $x = OM$, the tangents intersect at a point F on the $y$-axis,

$$\text{and} \qquad OF = -2OM^3$$
$$= -2MP_2.$$
$$\text{Hence} \qquad FH = 2HP_2,$$
$$\text{and} \qquad OH = 2HM.$$

We have thus a method of drawing the tangents at $P_1$, $P_2$ and $P_3$.

Take $MH = \frac{1}{3}MO$; join $P_2H$ and produce it to cut $Oy$ in F; join $FP_1$ and $FP_3$.

(3) It may be easily shown that the area of the triangle $P_1FP_3 = OM^2$; and that the area bounded by OM, $MP_2$ and the curve $y = x^3$, namely, $\int_0^x x^3 dx = \frac{3}{4}$ of the rectangle $MP_2 . MH$.

(4) If $\phi_1$, $\phi_2$, $\phi_3$ be the angles made with $Ox$ by $P_1F$, $P_2F$, and $P_3F$, and $\psi$ the $\angle P_1FP_3$, we have

$$\psi = \phi_3 - \phi_1, \quad \tan\phi_1 = 3x^2 - 1,$$
$$\tan\phi_2 = 3x^2 \quad ,$$
$$\tan\phi_3 = 3x^2 + 1,$$

from which we find

$$\tan\psi = \frac{2}{9x^4} = \frac{2}{\tan^2\phi_2}.$$

FIGURE 3.

§4. $\mu_1$, $\mu$ are turning-points on the curve $y = x^3 - x$; their coordinates are

$$\pm\frac{1}{\sqrt{3}}, \ \mp\frac{2}{3\sqrt{3}}, \text{ or } \pm 0{\cdot}57735, \ \mp 0{\cdot}38490.$$

The tangent at $\mu_1$ is parallel to $Ox$, hence for this point the angle $P_1FP_3$ (see last figure) is $\tan^{-1}2$; and therefore $\tan\phi_2 = 1$, that is, the tangent to $y = x^3$ makes an angle of 45° with $Ox$.

If in the equation $x^3 - x = A$,

$$A = P\mu = 0{\cdot}38490 \ldots,$$

the roots are OP, OP and OP′;

now
$$OP = -\frac{1}{\sqrt{3}},$$

and since the sum of the roots is zero,

$$\therefore \ OP' = \frac{2}{\sqrt{3}}.$$

The *Irreducible Case* corresponds to the region of the curve between the two points $\mu'$ and $\mu_1'$ whose coordinates are

$$\pm\frac{2}{\sqrt{3}}, \ \pm\frac{2}{3\sqrt{3}}, \text{ that is, } \pm 1{\cdot}1547, \ \pm 0{\cdot}385.$$

§5. Tables may easily be drawn up giving the values of $x^3 - x$, $x^3$ and $x^3 + x$ for a certain number of values of $x$; and from these tables we can find values of $x$ between which the roots of the cubic must lie.

Taking, for example, the cubic

$$x^3 - x = \frac{1}{\sqrt{7}} = 0\cdot378\ldots,$$

since $\dfrac{1}{\sqrt{7}}$ is less than $\dfrac{2}{3\sqrt{3}}$, that is, $0\cdot385$, (see § 4)

the given equation must have 3 real roots, one positive and two negative; from a table giving the values of $x^3 - x$ for values of $x$ between 0 and 1·5 differing by ·1, (or from the curve $y = x^3 - x$ itself) we find that one root lies between

$$x = -0\cdot6 \text{ and } x = -0\cdot7,$$

since for $x = 0\cdot6,\ x^3 - x = -0\cdot384$

and for $x = 0\cdot7,\ x^3 - x = -0\cdot357$;

a second root lies between

$$x = -0\cdot5 \text{ and } x = -0\cdot6,$$

since for $x = 0\cdot5,\ x^3 - x = -0\cdot375$

and for $x = 0\cdot6,\ x^3 - x = -0\cdot384$;

the third root lies between

$$x = 1\cdot1 \text{ and } x = 1\cdot2.$$

Closer approximations to the roots of the cubic can now be found in several ways.

(1) By completing the table of values of $x^3 \pm x$ for values of $x$ lying between the limiting values obtained for the roots,

(2) or by a special method, such as the following (Newton's method).

Take, for example, the equation

$$x^3 + x = A;$$

suppose A so great that $x$ is small compared with $x^3$; a first approximation to $x$ will thus be got by taking $x^3 = A$, which gives

$$x = \sqrt[3]{A} = a, \text{ say.}$$

FIGURE 4.

Let $OP = a$, then $PM = a^3 = A$,

and $PN = a^3 + a$;

the root sought is OR, where $RL = PM = A$.

Draw NB a tangent at N to $y = x^3 + x$;
let $x_1 = OP_1 =$ abscissa of B.

$$\tan NBM = 3a^2 + 1,$$

$$\therefore \quad BM = \frac{a}{3a^2+1};$$

$$\therefore \quad x_1 = OP_1 = a - \frac{a}{3a^2+1} = \frac{3a^3}{3a^2+1}.$$

This is a closer approximation to the root.

Repeat the process by drawing the tangent at B′, the point in which $P_1B$ meets the curve $y = x^3 + x$; let it cut ML at B″ and let $x_2 = OP_2$ be abscissa of B″.

It can easily be shown that

$$x_2 = \frac{2x_1^3 + a^3}{3x_1^2 + 1};$$

$x_2$ is a third approximation to the root.

Similarly we find

$$x_3 = \frac{2x_2^3 + a^3}{3x_2^2 + 1}$$

$$\ldots\ldots \quad \ldots\ldots\ldots$$

$$x_{n+1} = (2x_n^3 + a^3)/(3x_n^2 + 1).$$

A similar method can be applied to the equation

$$x^3 - x = A$$

and gives the following approximations to $x$:—

$$x = \sqrt[3]{A} = a, \text{ say},$$

$$x_1 = \frac{3a^3}{3a^2 - 1},$$

$$x_2 = \frac{2x_1^3 + a^3}{3x_1^2 - 1},$$

$$\ldots\ldots$$

$$x_{n+1} = \frac{2x_n^3 + a^3}{3x_n^2 - 1}.$$

*Example*:—To find approximations to the roots of the equation

$$x^3 - 2x - 5 = 0.$$

Put $\quad \theta x' = x;$

re get $$\theta^2 x'^3 - 2\theta x' - 5 = 0,$$

that is, $$x'^3 - \frac{2}{\theta^2}x' = \frac{5}{\theta^3}.$$

Choosing $$\theta = \sqrt{2},$$

re get $$x'^3 - x' = \frac{5}{2\sqrt{2}} = 1{\cdot}7678.$$

$$1{\cdot}7678 \text{ is } > \frac{2}{3\sqrt{3}};$$

he equation has therefore only one real root and it is positive.

From the table of values of $x^3 - x$ we find that the root lies etween 1·4 and 1·5; completing the table for values of $x$ between ·4 and 1·5 we find that $x' = 1{\cdot}48$ to the nearest hundredth.

By applying the method explained above we find the following loser approximations to the root

$$1{\cdot}481, \quad 1{\cdot}4811, \quad 1{\cdot}48107.$$

Taking $$x' = 1{\cdot}48107$$

re get $$x = \sqrt{2} \times 1{\cdot}48107$$
$$= 2{\cdot}09455.$$

Closer approximations can easily be found.

Lagrange and Newton, both of whom solved this equation, give

$$x = 2{\cdot}09455147.$$

When the only real root $a$ of a cubic $x^3 - x - A = 0$ has been ound, the two imaginary roots can be deduced.

Let them be $\alpha \pm i\beta$.

The sum of the three roots being zero, we have

$$a + 2\alpha = 0;$$

$$\therefore \quad \alpha = -\frac{a}{2};$$

he product of the roots being A, we have

$$a(\alpha^2 + \beta^2) = A,$$

nd $\therefore$ $$\beta = \pm\sqrt{\frac{A}{a} - \alpha^2} = \sqrt{\frac{A}{a} - \frac{a^2}{4}}.$$

§ 6. *Equations of higher degree than the 3rd.*

An equation of the 4th degree may be reduced to one of the forms

$$x^4 + x^2 + px + q = 0,$$
$$x^4 \quad\quad + px + q = 0,$$
$$x^4 - x^2 + px + q = 0,$$

so that the real roots may be found by drawing the curves $y = x^4 \pm x^2$ and $y = x^4$ and the straight line $y = -(px + q)$.

The equation of the 5th degree can be brought to one of the three forms

$$x^5 - x^3 + px^2 + qx + r = 0,$$
$$x^5 + x^3 + px^2 + qx + r = 0,$$
$$x^5 \quad\quad + px^2 + qx + r = 0.$$

The real roots will be the abscissæ of the points of intersection of one of the curves

$$y = x^5 - x^3, \quad y = x^5 + x^3, \quad y = x^5$$

and the parabola

$$y = -(px^2 + qx + r).$$

In general the problem of finding the real roots of an equation of the $m$th degree is reduced to that of finding the points of intersection of a curve of degree $m - 3$ and one of the curves

$$y = x^m - x^{m-2}, \quad y = x^m, \quad y = x^m + x^{m-2}.$$

§ 7. *Discussion of the curves*

$$y = x^m - x^{m-2}, \quad y = x^m, \quad y = x^m + x^{m-2}.$$

(1) If $y_1$, $y_2$, $y_3$ be the 3 ordinates for the same abscissa,

$$2y_2 = y_1 + y_3,$$

$$2\frac{dy_2}{dx} = \frac{dy_1}{dx} + \frac{dy_3}{dx},$$

and $$2\int_0^x y_2 dx = \int_0^x y_1 dx + \int_0^x y_3 dx.$$

(2) If $x = 1$, $y_1 = 0$ and $\frac{dy_1}{dx} = 2$; this value of $\frac{dy_1}{dx}$ is independent of $m$, so that the curves $y = x^m - x^{m-2}$, for all values of $m$, have a common tangent at the point (1, 0).

(3) At the origin, if $m>3$, the curves touch the axis of $x$; if ı is even the three curves are symmetrical with respect to $Oy$; ' $m$ is odd the curves are symmetrical with respect to the origin 'hich must therefore be a point of inflexion on each of the curves; ıe radius of curvature is infinite at the origin if $m>2$.

(4) For the curve $y=x^m-x^{m-2}$

$$\frac{dy}{dx}=0 \text{ where } mx^{m-1}-(m-2)x^{m-3}=0.$$

'his equation has $m-3$ roots equal to 0, which define the point of ɔntact of the curve and $Ox$ (if $m>3$); and two other roots $=\pm\sqrt{\frac{m-2}{m}}$; at the points whose abscissæ are $\pm\sqrt{\frac{m-2}{m}}$ the ıngent to the curve is parallel to $Ox$; the value of the ordinate at hese points is given by

$$y=\left(\frac{m-2}{m}\right)^{\frac{m}{2}}-\left(\frac{m-2}{m}\right)^{\frac{m}{2}-1}.$$

For the locus of the points at which the tangents to the curves $=x^m-x^{m-2}$ (for different values of $m$) are parallel to $Ox$ we have he equation

$$y=x^{\frac{2}{1-x^2}}-x^{\frac{2x^2}{1-x^2}}.$$

(5) It may be shown as in the case of the curves

$$y=x^3-x, \quad y=x^3, \quad y=x^3+x$$

hat the tangents to the curves

$$y=x^m-x^{m-2}, \quad y=x^m, \quad y=x^m+x^{m-2}$$

t points which have the same abscissa, are concurrent, intersecting t

$$\left(\frac{m-3}{m-2}x, \; -\frac{2x^m}{m-2}\right);$$

he locus of this point for different values of $x$ is the curve

$$y=-\frac{2(m-2)^{m-1}}{(m-3)^m}x^m.$$

If $m=3$, this equation becomes $x=0$ (the particular case already .oticed).

For the curve $y=x^m$, the subtangent is $\frac{x}{m}$, a given fraction

of the abscissa. This leads to a method of drawing the tangents to the three curves at points which have a common abscissa.

Let OP be the abscissa and $M_1$, $M_2$, $M_3$ the three points in which the ordinate through P cuts the curves. Take PG (along $xO$) $= \dfrac{OP}{m}$ and join $GM_2$; this line will touch $y = x^m$.

Let F be the point of intersection of the tangents and OH its abscissa; then $OH = \dfrac{m-3}{m-2}x = \dfrac{m-3}{m-2}OP$ and $PH = \dfrac{OP}{m-2}$; besides $\dfrac{M_2F}{M_2G} = \dfrac{PH}{PG} = \dfrac{m}{m-2}$; so that to get F, produce $M_2G$ a distance $= \dfrac{2}{m-2}M_2G$; then join $FM_1$, $FM_3$; these lines touch the curves $y = x^m - x^{m-2}$ and $y = x^m + x^{m-2}$.

The area of the triangle $M_1FM_3 = \frac{1}{2}M_1M_3 \,.\, HP$

$$= \frac{x^{m-1}}{m-2},$$

$$= \frac{m-1}{m-2}\int_0^x x^{m-2}dx.$$

(6) *Radius of curvature.*

For the curve $y = x^m$ we find

$$\rho = \frac{\left(1 + m^2\dfrac{y^2}{x^2}\right)^{3/2}}{m(m-1)\dfrac{y}{x^2}}.$$

FIGURE 5.

Let M be a point on the curve, MG the tangent at M, MP the ordinate, MN the normal.

$$GP = \frac{x}{m};$$

$$\therefore \quad 1 + m^2\frac{y^2}{x^2} = 1 + \frac{MP^2}{GP^2} = \frac{GM^2}{GP^2};$$

$$\therefore \quad \rho = \frac{m}{m-1}\,\frac{GM^3}{GP.PM}.$$

Draw GR, $\perp$ to MG to meet MP produced at R.

Then $$GM^2 = MR . MP;$$

$$\therefore \quad \rho = \frac{m}{m-1} \frac{MR.GM}{GP}$$

$$= \frac{MR}{\cos\beta} \times \frac{m}{m-1}; \text{ (where } \beta = \angle MGx).$$

Draw RS parallel to $Ox$ to meet MN at S.

Then $$\rho = \frac{MR}{\cos\beta} \times \frac{m}{m-1} = MS \times \frac{m}{m-1}.$$

Hence to find C the centre of curvature, produce NM to C so that $MC = \dfrac{m}{m-1} \times MS.$

In the case of the parabola $y = x^2$, $\rho = 2MS$, so that S lies on the directrix ($y = -\frac{1}{4}$) of the parabola.

§ 8. To construct the curve $y = xf(x)$ from the curve $y = f(x)$.

Figure 6.

Let OM be the curve $y = f(x)$, M a point on it whose abscissa is OP.

Take, in the direction XO, a length $PQ = 1$.

Join MQ. Draw ON parallel to QM to meet PM at N.

Then $$\frac{PN}{PM} = \frac{PO}{PQ};$$

$$\therefore \quad PN = \frac{PO . PM}{PQ} = xf(x).$$

Hence N is a point on the curve $y = xf(x)$.

If we can draw the tangent at $(x, f(x))$ it will be possible to draw the tangent at $(x, xf(x))$.

Let $z = xf(x)$ be the equation of the curve constructed from $y = f(x)$.

Putting $z = xy$, we get

$$\frac{dz}{dx} = y + x\frac{dy}{dx},$$

that is, $$\tan\beta = y + x\tan\alpha$$

where $$\tan\beta = \frac{dz}{dx} \text{ and } \tan\alpha = \frac{dy}{dx}.$$

Hence if OS be drawn parallel to MR, the tangent at M to $y=f(x)$, to cut PM produced at S and SS′ be taken $=$ PM, PS′ will $=y+x\tan\alpha$; if S′Q be joined, since PQ $=1$, S′Q will be parallel to the tangent at N to the curve $z=xf(x)$. The tangent required is $\therefore$ NT, drawn parallel to S′Q.

§ 9. *Equations of higher degree than the 5th.*

To find the real roots of an equation of the $m$th degree we have to find the points of intersection of one of the curves

$$y=x^m-x^{m-2},\quad y=x^m,\quad y=x^m+x^{m-2}$$

and of a parabolic curve of degree $m-3$ at most.

A combination of this method and of the following will often simplify the solution. It consists in substituting for the curve whose equation is

$$y=\mathrm{A}x^m+\mathrm{B}x^{m-1}+\ldots+\mathrm{G}x^{m-n}+\mathrm{H}x^{m-n-1}+\ldots+\mathrm{P}$$

the curve whose equation is

$$z=\frac{x^{m-n}(\mathrm{A}x^n+\mathrm{B}x^{n-1}+\ldots+\mathrm{G})}{\mathrm{H}x^{m-n-1}+\ldots+\mathrm{P}}$$

and drawing the straight line $z=-1$, to cut it.

The solution of an equation of degree $m$ may thus be reduced to the solution of equations of much lower degrees.

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## On the Teaching of Geometry.

By John Turner, M.A., B.Sc.

*Third Meeting,* 12*th January* 1906.

D. C. M'INTOSH, Esq., M.A., President, in the Chair.

## On a Nine-Point Conic, etc.

By P. PINKERTON, M.A.

1. The Pole and Polar Theorem for the Circle may be proved as ollows :—

Let P be any point, and C the centre of a circle.

Let AB be the diameter through P, and QQ′ any chord passing hrough P.

Let AQ, BQ′ meet in M; AQ′, BQ in O; MO, AB in N; and MO, QQ′ in P′.

Then, by the harmonic properties of a complete quadrilateral,

N is the harmonic conjugate of P with respect to A, B; and

P′ is the harmonic conjugate of P with respect to Q, Q′.

N is therefore a fixed point. And NP′ is perpendicular to AB, ince O is clearly the orthocentre of triangle MAB. But P′ is *any* oint on the polar of P; hence the polar of P is the perpendicular o the diameter of the circle which passes through P, drawn through he harmonic conjugate of P with respect to the extremities of that iameter. *

2. The proof does not hold for any conic, as O is the orthocentre f triangle MAB only when the conic is a circle. The proof, howver, could be made general by using the following theorem, of which he theorem regarding the concurrence of the perpendiculars of a riangle is a particular case.

*Theorem*: If ABC is any triangle and if AD, BE, CF are lines rawn through the vertices such that AD, BC; BE, CA; CF, AB re parallel to pairs of conjugate diameters of a fixed conic, then AD, BE, CF are concurrent.

* This proof was given to me by R. VICKERS, one of my pupils.—P.P.

For, let BE and CF meet at O. Join AO.

Then $$\frac{\sin BAO}{\sin CAO} \cdot \frac{\sin CBO}{\sin ABO} \cdot \frac{\sin ACO}{\sin BCO} = -1;$$

$$\therefore \quad \frac{\sin(\alpha' - \gamma)}{\sin(\alpha' - \beta)} \cdot \frac{\sin(\beta' - \alpha)}{\sin(\beta' - \gamma)} \cdot \frac{\sin(\gamma' - \beta)}{\sin(\gamma' - \alpha)} = 1$$

where the lines BC, CA, AB, OA, OB, OC make, with the $x$-axis of a rectangular system of reference, angles $\alpha$, $\beta$, $\gamma$, $\alpha'$, $\beta'$, $\gamma'$.

Hence $$\frac{\tan\alpha' - \tan\gamma}{\tan\alpha' - \tan\beta} \cdot \frac{\tan\beta' - \tan\alpha}{\tan\beta' - \tan\gamma} \cdot \frac{\tan\gamma' - \tan\beta}{\tan\gamma' - \tan\alpha} = 1;$$

$$\therefore \quad (l' - n)(m' - l)(n' - m) = (l' - m)(m' - n)(n' - l)$$

where $\qquad l = \tan\alpha, \; l' = \tan\alpha'$, etc.

$$\therefore \; ll'(m + m' - n - n') + (l + l')(nn' - mm') + (n + n')mm' - (m + m')nn' = 0.$$

But with the usual notation

$$a + h(m + m') + bmm' = 0$$

$$\text{and} \quad a + h(n + n') + bnn' = 0;$$

$$\therefore \; m + m' - n - n' : nn' - mm' : (n + n')mm' - (m + m')nn' = b : h : a,$$

whence $\qquad a + h(l + l') + bll' = 0,$

*i.e.*, AO and BC are parallel to a pair of conjugate diameters of the conic. This proves the theorem.

3. The Pole and Polar Theorem for a conic follows at once by the method of § 1. For if the circle is replaced by a conic, AQ′ and BQ′ are supplementary chords and therefore parallel to a pair of conjugate diameters of the conic; so also are AQ and BQ; therefore so also are NP′ and AB. Hence the theorem.

4. The Theorem of § 2 leads to a proposition in conics, corresponding to a fundamental property of the orthocentre of a triangle, viz.:

If ABC is a triangle whose vertices lie on a conic, whose centre is O, and if the concurrent lines AH, BH, CH be drawn so that AH, BC; BH, CA; CH, AB are parallel to pairs of conjugate diameters of the conic, then AH = 2OL, etc., where L, M, N are the middle points of BC, CA, AB.

Following the proof for the corresponding property of the orthocentre, draw the diameter COZ, and join BZ, AZ.

Then ZB, BC are parallel to conjugate diameters of the conic, since they are supplementary chords.

$$\therefore \text{ ZB is parallel to AH,}$$
$$\text{similarly ZA is parallel to BH;}$$
$$\therefore \text{ ZB} = \text{AH},$$
$$\text{and ZB} = 2\text{OL},$$
$$\therefore \text{ AH} = 2\text{OL}.$$

5. Hence the following, corresponding to the theorem of the nine-point circle.

Let AH, BH, CH meet BC, CA, AB in D, E, F; and let P, Q, R be the middle points of AH, BH, CH; then the nine points L, M, N, P, Q, R, D, E, F lie on a conic whose centre is the middle point of OH.

Take X the middle point of OH.

A conic is determined if we know its centre and three points on it. Consider the conic whose centre is X and which passes through L, M, N.

Now AH = 2OL in magnitude and direction by § 4,

$\therefore$ PH = OL in magnitude and direction,

$\therefore$ X is the middle point of LP,

$\therefore$ P lies on the conic; similarly for Q, R.

Again OL, being the diameter conjugate to BC, is parallel to HD; and the middle point of MN is the middle point of AL, therefore the join of the middle points of LD and MN is parallel to HD or OL and so bisects OH.

Remembering that MN is parallel to LD, we see that D lies on the conic. Similarly E, F lie on the conic. This completes the proof.

## The Parabolic Path of a Projectile.

The proof, usually given, that the path of a projectile *in vacuo* is a parabola, assumes the equivalent of the equation to a parabola referred to a tangent and the diameter through the point of contact as axes.

The following proof * requires only the theorem, $PN^2 = 4AS.AN$.

Let A be the highest point of the path of the projectile.

Let P be the position of the projectile at time $t$ after the moment of projection; $V\cos\alpha$, $V\sin\alpha$ the horizontal and vertical components of the initial velocity.

Let a vertical line through A and a horizontal line through P meet in N.

Then

vertical component of velocity at time $t = V\sin\alpha - gt$,

and time required to travel the path $AP = \dfrac{V\sin\alpha - gt}{g}$;

$$\therefore \quad PN = \frac{V\cos\alpha(V\sin\alpha - gt)}{g}$$

$$\text{and} \quad AN = \frac{(V\sin\alpha - gt)^2}{2g};$$

$$\therefore \quad PN^2 = \frac{2V^2\cos^2\alpha}{g}.AN. \qquad \therefore \text{ etc.}$$

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* Given to me by J. G. GIBSON, one of my pupils.—P.P.

## The determination of the centre of gravity of a circular arc by reference to a principle of Dynamics.

By G. E. CRAWFORD, M.A.

Mr H. Poincaré, in a recent work (*la Valeur de la Science*) peaks of the interest attaching to proofs of theorems in pure eometry by arguments derived from mechanical or physical conideration. The following is a case in point.

To find the centre of gravity of a circular arc by reference to a rinciple of dynamics.

Let ABC be a material circular arc rotating freely about its entre O under no forces (Fig. 7).

Then constants are

$\omega$, the uniform angular velocity about O,

$r$, the radius of the circle,

T, the tension anywhere,

$\rho$, the line density.

Choose any arc AB to isolate mentally and consider its motion. he non-constants are

$2\alpha$, the angle AB subtends at O,

$x$, the required distance from O to G the centre of gravity of the arc.

Then we have the equation of acceleration along GO,

$$(2\alpha r\rho)\omega^2 x = 2\mathrm{T}\sin\alpha, \quad \text{or} \quad \frac{x\alpha}{\sin\alpha} = \frac{\mathrm{T}}{\omega^2 r\rho}.$$

We can now take any other arc in which $y$, $\beta$ correspond to $x$, $\alpha$ espectively and we get $\dfrac{y\beta}{\sin\beta} = \dfrac{\mathrm{T}}{\omega^2 r\rho}$ as before.

$$\therefore \quad \frac{x\alpha}{\sin\alpha} = \frac{y\beta}{\sin\beta}.$$

ut $\beta$ can be chosen to vanish, in which case $y$ becomes equal to $r$ nd $\dfrac{\beta}{\sin\beta}$ to unity,

$$\therefore \quad x = \frac{r.\sin\alpha}{\alpha}. \qquad \text{Q.E.D.}$$

# On the Reduction of $\int \frac{(\mathrm{L}x+\mathrm{M})dx}{(\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C})^m\sqrt{ax^2+2bx+c}}$.

By D. K. Picken, M.A.

The simplest type of irrational algebraic function of $x$ is gi[ven] by $f(x, y)$ a rational function of $x$ and $y$, in which $y$ is a varia[ble] whose dependence on $x$ is determined by the equation $\phi(x, y) =$ where $\phi(x, y)$ is a polynomial of the second degree in $x$ and $y$.

It is clear that the function can be expressed in the real form

$$\psi(x)+\Sigma\left\{\mathrm{A}x^m+\frac{\mathrm{B}}{(x-p)^m}+\frac{\mathrm{L}x+\mathrm{M}}{(\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C})^m}\right\}\frac{1}{\sqrt{ax^2+2bx+c}}$$

where $\psi(x)$ is a rational function of $x$; and for the integration have to obtain formulæ of reduction for

$$\text{(i)}\ \int\frac{x^m dx}{\sqrt{ax^2+2bx+c}},\qquad \text{(ii)}\ \int\frac{dx}{(x-p)^m\sqrt{ax^2+2bx+c}},$$

and (iii) $\int\frac{(\mathrm{L}x+\mathrm{M})dx}{(\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C})^m\sqrt{ax^2+2bx+c}}$ (where $\mathrm{B}^2-\mathrm{AC}$ is negati[ve]

The method of integration by parts can be simply applied to cases (i) and (ii) and we get reduction formulæ, which may regarded as the identities obtained by differentiating the functio[ns]

$$x^{m-1}\sqrt{ax^2+2bx+c}\quad\text{and}\quad\frac{\sqrt{ax^2+2bx+c}}{(x-p)^{m-1}}.$$

The use of this method is not quite so obvious in case (iii); the purpose of this note is to show that reduction formulæ can obtained by differentiations similar to those in cases (i) and (ii).

If $\quad I_m = \int \frac{dx}{(Ax^2+2Bx+C)^m \sqrt{ax^2+2bx+c}} \equiv \int \frac{dx}{S^m \sqrt{R}},$

$$I_m' = \int \frac{x dx}{S^m \sqrt{R}}, \quad P_m = \frac{\sqrt{R}}{S^{m-1}}, \quad P_m' = \frac{x\sqrt{R}}{S^{m-1}};$$

hen

$$\frac{d}{dx}(P_m) = \frac{ax+b}{S^{m-1}\sqrt{R}} - 2(m-1)\,.\,\frac{(Ax+B)(ax^2+2bx+c)}{S^m\sqrt{R}}$$

$$= \frac{ax+b}{S^{m-1}\sqrt{R}} - 2(m-1)\,.\,\frac{(ax+p_1)S+q_1x+r_1}{S^m\sqrt{R}},$$

here $p_1$, $q_1$, $r_1$ are constants obtained from the identity

$$(ax+p_1)(Ax^2+2Bx+C)+q_1x+r_1 \equiv (Ax+B)(ax^2+2bx+c).$$

ence we can write

$$\frac{d}{dx}(P_m) = \frac{A_m x + B_m}{S^{m-1}\sqrt{R}} + \frac{C_m x + D_m}{S^m\sqrt{R}},$$

here $A_m$, $B_m$, $C_m$, $D_m$ are determinate constants; and, therefore,

$$[P_m] = A_m\,.\,I_{m-1}' + B_m\,.\,I_{m-1} + C_m\,.\,I_m' + D_m\,.\,I_m \ldots\ldots (m).$$

Similarly,

$$\frac{d}{dx}(P_m') = \frac{ax^2+2bx+c+x(ax+b)}{S^{m-1}\sqrt{R}} - 2(m-1)\frac{x(Ax+B)(ax^2+2bx+c)}{S^m\sqrt{R}}$$

$$= \frac{\frac{2a}{A}S + p_2x + q_2}{S^{m-1}\sqrt{R}} - 2(m-1)\frac{\frac{a}{A}S^2 + (p_3x+q_3)S + r_3x + s_3}{S^m\sqrt{R}},$$

here the coefficients $p_2$, $q_2$, $p_3$, $q_3$, $r_3$, $s_3$ are determinate as above.

hus $[P_m'] = E_m\,.\,I_{m-2} + F_m\,.\,I_{m-1}' + G_m I_{m-1} + H_m I_m' + K_m\,.\,I_m \ldots (m)'$.

The equations $(m)$ and $(m)'$ give $I_m$ and $I_m'$ in terms of $_{m-1}$, $I_{m-1}'$, $I_{m-2}$; thus from the equations (2) and (2)', which form he last pair of the system, we get $I_2$, $I_2'$ in terms of $I_1$, $I_1'$ and $I_0$, nd finally we express $I_m$ and $I_m'$ in terms of $I_1$, $I_1'$ and $I_0$.

(In numerical cases the ordinary algebraic devices for obtaining he numbers $p_1$, $q_1$, ... $r_3$, $s_3$ would, of course, be employed.)

*Fourth Meeting, 9th February* 1906.

D. C. M'Intosh, Esq., M.A., President, in the Chair.

## On the relations of certain conics to a triangle.

By W. G. Fraser, M.A.

If we join the angular points ABC of a triangle to any poin the locus of the centres of conics passing through A, B, C, O i conic bisecting the six joins of the four points and passing throu the intersections of OA, BC; OB, AC; and OC, AB. This conic analogous to the nine-point circle, and at last meeting of the Soci Mr Pinkerton showed that its centre lies on the line joining O the centroid. In what follows an attempt is made still further generalise this conception.

Figure 8.

Take any two points O and O′, and let OA, OB, OC m BC, CA, AB in P, Q, R, with a corresponding construction wi regard to O′. Then the six points P, Q, R, P′, Q′, R′ lie on a co

If $\xi$, $\eta$, $\zeta$ and $\xi'$, $\eta'$, $\zeta'$ be the areal coordinates of O and referred to the triangle ABC, the equation of the conic is

$$x^2/\xi\xi' + y^2/\eta\eta' + z^2/\zeta\zeta' - yz(1/\eta\zeta' + 1/\eta'\zeta) - zx(1/\zeta\xi' + 1/\zeta'\xi)$$
$$- xy(1/\xi\eta' + 1/\xi'\eta) = 0.$$

If this conic cut OA, OB, etc., in L, M, etc., then at L we h $y/\eta = z/\zeta$, whence

$$x^2/\xi\xi' - x\{\zeta(1/\zeta\xi' + 1/\zeta'\xi) + \eta(1/\xi\eta' + 1/\xi'\eta)\} = 0.$$

The root $x = 0$ corresponds to P; hence, at L

$$x = 2\xi + \xi'(\eta/\eta' + \zeta/\zeta')$$
$$= \xi + \xi'(\xi/\xi' + \eta/\eta' + \zeta/\zeta'), \text{ with } y = \eta,\ z = \zeta.$$

Now the coordinates of O are $(\xi, \eta, \zeta)$, and of A, (1, 0, 0). Hence, if $\mathrm{OL}/\mathrm{LA} = \kappa_1$, the coordinates of L are proportional to $\xi + \kappa_1, \eta, \zeta$, so that

$$\kappa_1 = \xi'(\xi/\xi' + \eta/\eta' + \zeta/\zeta') \qquad - \quad - \quad (1)$$

with similar results for M, N, L′, M′, N′. This may be thrown into the form

$$\frac{\mathrm{OL}}{\mathrm{LA}} = \frac{\Delta\mathrm{BO'C}}{\Delta\mathrm{ABC}}\left\{\frac{\Delta\mathrm{BOC}}{\Delta\mathrm{BO'C}} + \frac{\Delta\mathrm{COA}}{\Delta\mathrm{CO'A}} + \frac{\Delta\mathrm{AOB}}{\Delta\mathrm{AO'B}}\right\}.$$

If we write $p, q, r$ for $1/\xi, 1/\eta, 1/\zeta$, the equation of the conic becomes

$$pp'x^2 + qq'y^2 + rr'z^2 - (qr' + q'r)yz - (rp' + r'p)zx - (pq' + p'q)xy = 0. \quad (2)$$

This conic, the self-conjugate conic

$$pp'x^2 + qq'y^2 + rr'z^2 = 0 \qquad - \quad - \quad (3)$$

and the circum-conic

$$(qr' + q'r)yz + (rp' + r'p)zx + (pq' + p'q)xy = 0 \quad (4)$$

have four points in common. Another conic through the four points is

$$pp'x^2 + qq'y^2 + rr'z^2 + (qr' + q'r)yz + (rp' + r'p)zx + (pq' + p'q)xy = 0$$

$$\text{or} \quad (px + qy + rz)(p'x + q'y + r'z) = 0$$

which represents the "trilinear polars" of O and O′.

If we say that the conic (2) is a conic "concurrently connected" with the triangle, and that it is "determined" by the points O and O′, we see that

If a concurrently-connected conic meet the trilinear polars of its determining points in X, Y, X′, Y′, then the family of conics passing through X, Y, X′, Y′ will include a self-polar conic and a circumconic.

These three conics are generalisations of the nine-point, self-polar, and circum-circles, for which two of the common points are the circular points, and the other two lie on the "orthic axis," which is their radical axis.

We may notice that the other two pairs of lines through X, Y, X′, Y′ will, through their trilinear poles, determine other two concurrently-connected conics of the same family.

The four-point family have a common self-conjugate triangle, one vertex of which is V, the intersection of XY and X′Y′. This point has therefore the same polar with respect to all the conics; now the coördinates of V are

$$qr' - q'r, \ rp' - r'p, \ pq' - p'q.$$

The polar of this for the self-conjugate conic (3) is

$$pp'(qr'-q'r)x+qq'(rp'-r'p)y+rr'(pq'-p'q)z=0$$

or $$(\eta\zeta'-\eta'\zeta)x+(\zeta\xi'-\zeta'\xi)y+(\xi\eta'-\xi'\eta)z=0$$

*i.e.*, the line OO′. Thus OO′ is the polar of V for every conic of the family, and reciprocally, the locus of the poles of any line through V is OO′ (together with the harmonic conjugate of the given line with respect to VX and VX′).

Now if V be at infinity we deduce that

If the trilinear polars of O and O′ be parallel, the centres of all the conics through X, Y, X′, Y′ lie on the line OO′ (together with the line midway between XY and X′Y′).

Furthermore, XY and X′Y′ are parallel to the diameters conjugate to OO′, so that OO′ bisects XY and X′Y′.

The analytical condition that XY and X′Y′ be parallel is

$$\begin{vmatrix} p & p' & 1 \\ q & q' & 1 \\ r & r' & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} \dfrac{1}{\xi} & \dfrac{1}{\xi'} & 1 \\ \dfrac{1}{\eta} & \dfrac{1}{\eta'} & 1 \\ \dfrac{1}{\zeta} & \dfrac{1}{\zeta'} & 1 \end{vmatrix}=0$$

showing that O, O′, and the centroid lie on a circum-conic.

Again, let X′Y′ be altogether at infinity. Then O′ coincides with the centroid G. By (1), the original concurrently-connected conic bisects OA, OB, OC, and becomes a nine-point conic; two of the common points of the family are at infinity, so that the conics are all homothetic; and their centres lie on OG. The equations of the nine-point, self-conjugate, and circum-conics of the family become

$$px^2+qy^2+rz^2-(q+r)yz-(r+p)zx-(p+q)xy=0 \quad - \quad (2')$$

$$px^2+qy^2+rz^2=0 \quad - \; - \; - \; - \; - \; - \; - \quad (3')$$

and $$(q+r)yz+(r+p)zx+(p+q)xy=0 \quad \cdot \; \cdot \; \cdot \; - \quad (4')$$

The centre, K, of (2′) is given by

$$2px_0-(p+q)y_0-(p+r)z_0=-(q+p)x_0+2qy_0-(q+r)z_0$$
$$=-(r+p)x_0-(r+q)y_0+2rz_0$$

whence
$$\frac{x_0}{2\xi+\eta+\zeta}=\frac{y_0}{\xi+2\eta+\zeta}=\frac{z_0}{\xi+\eta+2\zeta}.$$

Now if $x_0$, $y_0$, $z_0$ and $\xi$, $\eta$, $\zeta$ be the *actual* coordinates of K, and of O,

$$x_0+y_0+z_0=\xi+\eta+\zeta=1$$

whence
$$\frac{x_0}{\xi+1}=\frac{y_0}{\eta+1}=\frac{z_0}{\zeta+1}=\frac{1}{4};$$

$$\therefore\quad x_0=\frac{\xi+1}{4}=\frac{\xi+3\cdot\frac{1}{3}}{4},\quad y_0=\frac{\eta+3\cdot\frac{1}{3}}{4},\quad z_0=\frac{\zeta+3\cdot\frac{1}{3}}{4}.$$

But $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are the coordinates of G. Hence K divides OG internally in the ratio of 3 to 1.

The centre of (3′) is clearly at O; and the centre of (4′) may be shown to divide OG externally in the ratio of 3 to 1. The various centres are therefore relatively situated exactly as those of the circles which the conics become when O is the orthocentre.

## On the Pascal Hexagram.

By Prof. J. Jack.

§ 1. *Proof in case of the circle.*

Figure 9.

ABCDEF is any cyclic hexagon;

AB, DE meet in G,

BC, EF „ „ H,

CD, FA „ „ K; then G, H, K are in a straight line.

Draw KLNM parallel to BC and produce DE, EF, AB to meet it in L, N, M. Join DA, DM and let BC, DE meet in P.

$\angle$ PCD = $\angle$ BAD ∵ ABCD is a cyclic quadrilateral,

and $\angle$ PCD = $\angle$ DKM ∵ KM is parallel to CB;

∴ $\angle$ BAD = $\angle$ DKM; ∴ DAMK is a cyclic quadrilateral.

Again $\angle$ DEH = $\angle$ DAK ∵ AFED is cyclic;

and $\angle$ DMN, *i.e.*, $\angle$ DMK = $\angle$ DAK ∵ DAMK is cyclic;

∴ $\angle$ DEH = $\angle$ DMN and ∴ DMNE is cyclic.

Again $\dfrac{PB}{PE} = \dfrac{PD}{PC} = \dfrac{DL}{LK}$

and $\dfrac{PE}{PH} = \dfrac{LE}{LN} = \dfrac{LM}{LD}$; ∴ $\dfrac{PB}{PH} = \dfrac{LM}{LK}$;

∴ G, H, K are in a straight line.

Figures 10 and 11.

§ 2. O is any point on a chord AB of a conic; the focus S, the directrix and $e$ are given; the eccentric circle of O is described. Through O, radii O$a$, O$b$ are drawn parallel to SA, SB, in opposite

sense when O, S are on the *same* side of the directrix (Fig. 10) and in the same sense when on *opposite* sides (Fig. 11).

Then $ab$ passes through S.

For $$\frac{Oa}{OK\sin\theta}=\frac{SA}{AK\sin\theta};$$

$\therefore$ S, $a$, K are in a straight line. So S, $b$, K are in a straight line; $\therefore$ $ab$ goes through S.

§ 3. *Pascal's Theorem for the Conic (generally).*

ABCDEF is any hexagon inscribed in a conic.

AB, DE meet in $O_1$; BC, EF meet in $O_2$, and CD, FA meet in $O_3$.

Then $O_1$, $O_2$, $O_3$ are in a straight line.

The focus S, the directrix and $e$ are given.

Draw the eccentric circles of the points $O_1$, $O_2$, $O_3$ and draw in each of the circles the six radii parallel to SA, SB, SC, SD, SE and SF, in the opposite sense when O and S are on the same side of the directrix and in the same sense when O, S are on opposite sides of the directrix.

Let the radii be $O_1a_1, O_1b_1$, etc.; $O_2a_2$, $O_2b_2$, etc., etc.

Join the points $a_1b_1$, $b_1c_1$, $c_1d_1$, $d_1e_1$, $e_1f_1$, $f_1a_1$;

$a_2b_2$, $b_2c_2$, $c_2d_2$, etc.,

$a_3b_3$, $b_3c_3$, $c_3d_3$, etc.

Then the three cyclic hexagons $(abcdef)_1$, $(abcdef)_2$, $(abcdef)_3$ are similar and similarly situated.

Let $a_1b_1$, $d_1e_1$ meet in $g_1$,

$b_1c_1$, $e_1f_1$ „ „ $h_1$,

and $c_1d_1$, $f_1a_1$ „ „ $k_1$,

and similarly for the other two hexagons let the corresponding sides meet in $g_2$, $h_2$, $k_2$ and $g_3$, $h_3$, $k_3$.

Then $g_1h_1k_1$, $g_2h_2k_2$, $g_3h_3k_3$ are straight lines by the proof of the theorem in the case of a circle.

Now from the nature of the eccentric circle, $a_1b_1$, $d_1e_1$ meet in S, that is, the point $g_1$ is S.

Similarly „ „ $h_2$ is S

and „ „ $k_3$ is S.

Hence the three straight lines, $g_1h_1k_1$, $g_2h_2k_2$, $g_3h_3k_3$ have one point S common and they are parallel, because the figures are similar and similarly situated;

$\therefore$ the three lines are coincident.

Now taking the three triangles $O_1a_1g_1$, $O_2a_2g_2$, $O_3a_3g_3$ which are similar, we have

$$\frac{O_1g_1}{O_1a_1} = \frac{O_2g_2}{O_2a_2} = \frac{O_3g_3}{O_3a_3}$$

and if $O_1m_1$, $O_2m_2$, $O_3m_3$ are the perpendiculars from $O_1$, $O_2$, $O_3$ to the directrix, we have

$$\frac{O_1a_1}{O_1m_1} = \frac{O_2a_2}{O_2m_2} = \frac{O_3a_3}{O_3m_3} = e;$$

it therefore follows that

$$\frac{O_1g_1}{O_1m_1} = \frac{O_2g_2}{O_2m_2} = \frac{O_3g_3}{O_3m_3};$$

$\therefore$ $O_1$, $O_2$, $O_3$ are in a straight line, and it passes through the point in which the Pascal line of the cyclic hexagons meets the directrix.

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## On Newton's Theorem in the Calculus of Variations.

By J. H. MACLAGAN-WEDDERBURN, M.A.

*Fifth Meeting, 9th March* 1906.

D. C. M'INTOSH, Esq., M.A., President, in the Chair.

## Proofs of an Inequality.

By R. F. MUIRHEAD, M.A., D.Sc.

§ 1. The inequality of the Arithmetic and Geometric Means of $n$ positive quantities has been proved by many different methods; of which a classified summary has been given in the *Mathematical Gazette* (Vol. II., p. 283). The present article may be looked on as supplementary to that summary. It deals with proofs that belong to a general type, of which the proof given in the *Tutorial Algebra*, § 205, and that given by Mr G. E. Crawford in our *Proceedings*, Vol. XVIII., p. 2, are very special limiting cases. Proofs of the type in question consist of a finite number of steps, by which, starting from the $n$ given quantities, and changing two at a time according to some law, we reach a new set of quantities whose arithmetic mean is not greater, and whose geometric mean is not less than the corresponding means of the given quantities.

Mr Crawford's remark that his proof and that of the *Tutorial Algebra* are the only possible ones on the same lines is perhaps justifiable if the phrase "on the same lines" is very narrowly interpreted. But, as will be seen in what follows, there are an infinite number of possible proofs which have the general character above stated, and which share with the two referred to the "logical advantage that the number of mental steps in the process is finite." It is true that those two are the simplest of the kind, but I have thought it of interest to show that the others exist, though I do not suggest that they are so good.

§2. In order to simplify the exposition, I shall ... take no notice of special cases that may arise in which inequality reduces to equality. The proofs could be made more general by using the symbols $\not>$ and $\not<$ in place of $<$ and $>$ respectively, but the gain in generality would be at the expense of conciseness and clearness.

Let $a, b, c, \ldots$ be $n$ unequal positive quantities, of which A is the arithmetic, and G the geometric mean.

The following proof is only slightly more complex than that of Mr Crawford:—

Let $a$ be the greatest, $b$ the least of the given quantities.

Let $k$ be defined by the equation

$$G + k = a + b \qquad \text{(1)}$$

Then $\quad Gk - ab = G(a + b - G) - ab = (a - G)(G - b) > 0 \qquad \text{(2)}$*

Thus $\quad Gkcd \ldots > abcd \ldots > G^n$

$$\therefore \quad kcd \ldots > G^{n-1}. \qquad \text{(3)}$$

Let now $\quad G_1^{n-1} = kcd \ldots$, so that $G_1 > G. \qquad \text{(4)}$

Next treat the $n - 1$ quantities $k, c, d, \ldots$ as we treated $a, b, c, d, \ldots$, *i.e.*, Let $G_1 + k_1 =$ the sum of the greatest and least of them, say, $k$, $c$. It will follow, as before, that

$$G_1 k_1 def \ldots > kcdef \ldots$$
$$> G_1^{n-1}$$

$\therefore \quad k_1 def \ldots > G_1^{n-2}$, so that if $G_2^{n-2} \equiv k_1 def \ldots$

we have $$G_2 > G_1. \qquad \text{(5)}$$

Proceeding in this way, we finally get $G, G_1, G_2, \ldots G_{n-1}$, a set of quantities with the same sum as $a, b, c, d, e, \ldots$.

But $\quad G < G_1 < G_2 \ldots < G_{n-1}$. Hence $nG < a + b + c + d + \ldots \qquad \text{(6)}$

$$\therefore \quad G < A.$$

* Here we assume that G lies between $a$ and $b$, a fact which is easily proved.

Again, if we take $Ak = ab$, so that

$$A+k-a-b = A+\frac{ab}{A}-a-b = \frac{1}{A}(A-a)(A-b) < 0 \quad - \quad (7)$$

$$\therefore \quad A+k < a+b \quad - \quad - \quad (8)$$

$$\therefore \quad A+k+c+d+\ldots < a+b+c+d+\ldots < nA$$

$$\therefore \quad k+c+d+\ldots < (n-1)A. \quad - \quad (9)$$

Hence if $\quad (n-1)A_1 \equiv k+c+d+\ldots$, then $A_1 < A$. - - (10)

Proceeding in this manner, we get quantities $A, A_1, A_2, \ldots A_{n-1}$, whose product is $= abcd\ldots$, such that

$$A > A_1 > A_2 \ldots > A_{n-1}. \quad - \quad (11)$$

Hence

$$A^n > abcd\ldots$$

$$> G^n \quad - \quad (12)$$

$$\therefore \quad A > G.$$

§ 3. But we may generalise the method, by choosing $k$ so that A and $k$ may have in common with $a$ and $b$ not the value of their geometric or arithmetic mean, but the value of some mean which by its nature lies *between* the arithmetic and the geometric mean of two quantities.

For example, let $\quad \sqrt{A}+\sqrt{k} = \sqrt{a}+\sqrt{b} \quad$ - - - (13)

$$\therefore \quad \sqrt{A}\sqrt{k} - \sqrt{a}\sqrt{b}$$

$$= \sqrt{A}(\sqrt{a}+\sqrt{b}-\sqrt{A}) - \sqrt{a}\sqrt{b} = (\sqrt{a}-\sqrt{A})(\sqrt{A}-\sqrt{b}) > 0$$

$$\therefore \quad Ak > ab. \quad - \quad (14)$$

But by (13), $A+k+2\sqrt{A}\sqrt{k} = a+b+2\sqrt{a}\sqrt{b}$

$$\therefore \quad A+k < a+b \quad - \quad (15)$$

Thus $A, k, c, d, \ldots$ have a smaller sum and a greater product than $a, b, c, d, \ldots$.

Hence if $A_1$ is the arithmetic mean of $k, c, d, \ldots$, then $A_1 < A$.

Dealing with $k, c, d, \ldots$ as before with $a, b, c, d, \ldots$ and repeating the process, each time with one quantity fewer, we get

$$abcd\ldots\ldots < Akcd\ldots\ldots < AA_1k_1\ldots\ldots < AA_1A_2\ldots\ldots A_{n-1} < A^n$$

$$\therefore \quad G < A.$$

§ 4. The general type of such proofs may be indicated thus :—

Of $n$ positive quantities $a, b, c, d, \ldots$ let $a$ be one which is not less than any of the others - - - - - - - - (21)

and let A denote the arithmetic mean and G the geometric mean of $a, b, c, d, \ldots$.

Let us find M and $k$ such that

$$Mk \nless ab \qquad (22)$$

$$M + k \ngtr a + b \qquad (23)$$

Let us now re-arrange the quantities $k, c, d, \ldots$ and denote them by $a', b', c', \ldots$ where $a'$ is one which is not less than any of the others, and again find M′, $k'$ such that

$$M'k' \nless a'b' \qquad (24)$$

$$M' + k' \ngtr a' + b' \qquad (25)$$

and proceed to deal with $k', c', d', \ldots$ as previously with $k, c, d, \ldots$; and continue the process till the last M, say N, and the last $k$, say $l$, are found.

Then $$MM'M'' \ldots\ldots Nl \nless abcde \ldots\ldots \qquad (26)$$

and $$M + M' + M'' + \ldots + N + l \ngtr a + b + c + d + \ldots \qquad (27)$$

are deduced at once from (22), (24), (23), (25) and the other inequalities like them.

Now suppose further that the M's and $k$'s have been chosen so

that $$M \nless M' \nless M'' \ldots \nless N \nless l \ldots \qquad (28)$$

and we shall have, by (26) and (28), $M^n \nless abcd \ldots$

$$\therefore\ M \nless G. \qquad (29)$$

On the other hand, if we arrange so that

$$M \ngtr M' \ngtr M'' \ldots \ngtr N \ngtr l \qquad (30)$$

then by (27) we shall have $nM \ngtr a + b + c \ldots$

$$\therefore\ M \ngtr A. \qquad (31)$$

§ 5. Now in order to prove that $A \nless G$ in the case of (28) we need to secure that $M \ngtr A$, which would be most simply attained by putting $M = A$.

We must then choose the auxiliary quantity $k$ in such a manner as to satisfy (22) and (23); and for this it is in general sufficient to choose $k$ by assuming

$$\phi(M, k) = \phi(a, b) \qquad (32)$$

where $\phi(a, b)$ is some mean which by its nature is intermediate between $\sqrt{ab}$ and $(a + b)/2$.

As an example, let us take $\phi(a,\ b) \equiv \left\{\dfrac{a^2+pab+b^2}{p+2}\right\}^{\frac{1}{2}}$ where $p > 2$, so that

$$M^2+pMk+k^2 \quad = a^2+pab+b^2. \qquad (33)$$

Then $$(M+k)^2-(a+b)^2=(p-2)(ab-Mk). \qquad (34)$$

Thus $M+k-(a+b)$ and $ab-Mk$ have the same sign.

Again $(Mk-ab)(Mk+ab+pM^2)=M^2k^2-a^2b^2+pM^2(Mk-ab)$

$$\begin{aligned} &= M^2(k^2+pMk+M^2)-a^2b^2-M^4-pabM^2 \\ &= M^2(a^2+pab\ +b^2)\ -a^2b^2-M^4-pabM^2 \\ &= (a^2-M^2)(M^2-b^2). \qquad (35) \end{aligned}$$

Hence if we choose $b$ (as we may) to be the least of the quantities $a$, $b$, ......, then $a \nless M \nless b$, and (35) is $\nless 0$;

$$\therefore\ Mk\ \ \nless ab; \qquad (36)$$

$$\therefore\ M+k \ngtr a+b. \qquad (37)$$

Thus the conditions (22) and (23) are attained.

Again, $$\begin{aligned} nM &= a+b+c\ldots \\ &\nless M+k+c+d+\ldots, \quad \text{by (37)}, \\ \therefore\ \overline{n-1}M &\nless k+c+d+\ldots \\ &\nless a'+b'+c'+\ldots \\ &\nless \overline{n-1}M', \end{aligned}$$

since $M'$ is the arithmetic mean of $a'$, $b'$, $c'$, .... Hence $M \nless M'$, *i.e.*, the condition (28) is satisfied.

§ 6. It is now clear that the proposition $A \nless G$ is susceptible of proof in an infinite number of ways, each belonging to the type we are considering. This statement is already justified by the fact that each value of $p$ which we may choose, provided it is $< 2$, gives a separate variety of the type. It would perhaps be difficult to prove that the method of choosing $k$ indicated in connection with (32) will *always* satisfy the conditions required. But there is no difficulty in getting a large variety of forms of $\phi$ for which the proof is easily completed, still keeping $M \equiv A$.

As another example, take $\phi(a, b) \equiv \left(a^{\frac{1}{r}} + b^{\frac{1}{r}}\right)/2$, so that

$$M^{\frac{1}{r}} + k^{\frac{1}{r}} = a^{\frac{1}{r}} + b^{\frac{1}{r}}\,; \quad - \quad - \quad - \quad - \quad (38)$$

$$\therefore \quad M^{\frac{1}{r}} - b^{\frac{1}{r}} = a^{\frac{1}{r}} - k^{\frac{1}{r}} \quad - \quad - \quad - \quad - \quad (39)$$

and

$$\frac{M-b}{a-k} = \frac{M^{\frac{1}{r}} - b^{\frac{1}{r}}}{a^{\frac{1}{r}} - k^{\frac{1}{r}}} \cdot \frac{M^{\frac{r-1}{r}} + b^{\frac{1}{r}} M^{\frac{r-2}{r}} + \ldots\ldots + b^{\frac{r-1}{r}}}{a^{\frac{r-1}{r}} + k^{\frac{1}{r}} a^{\frac{r-2}{r}} + \ldots\ldots + k^{\frac{r-1}{r}}}. \quad - \quad (40)$$

Now (38) shows $k \nless b$ since $M \ngtr a$.

Hence each term of the numerator of the last fraction in (40) is $\ngtr$ the corresponding term in the denominator, while the fraction $\dfrac{M^{\frac{1}{r}} - b^{\frac{1}{r}}}{a^{\frac{1}{r}} - k^{\frac{1}{r}}} = 1$ in virtue of (39).

Hence $\quad \dfrac{M-b}{a-k} \ngtr 1\,; \quad \therefore \quad M - b \ngtr a - k\,;$

$$\therefore \quad M + k \ngtr a + b. \quad - \quad - \quad - \quad - \quad (41)$$

Again

$$M^{\frac{1}{r}} k^{\frac{1}{r}} - a^{\frac{1}{r}} b^{\frac{1}{r}} = M^{\frac{1}{r}}\left(a^{\frac{1}{r}} + b^{\frac{1}{r}} - M^{\frac{1}{r}}\right) - a^{\frac{1}{r}} b^{\frac{1}{r}}$$

$$= \left(a^{\frac{1}{r}} - M^{\frac{1}{r}}\right)\left(M^{\frac{1}{r}} - b^{\frac{1}{r}}\right) \nless 0\,;$$

$$\therefore \quad Mk \nless ab. \quad - \quad - \quad - \quad - \quad - \quad (42)$$

Thus the conditions (22) and (23) are fulfilled, and it is easy to prove, as before, that the condition (28) is also fulfilled.

Here again, $r$ being any positive integer, we have an infinite number of possible methods of proof.

§ 7. Going back now to the case (30) which is an alternative to (28), we should have to modify the method of proof. The proof requires that $M \nless G$, and the simplest way of securing this is to make $M \equiv G$. Then $k$ would have to be chosen so as to satisfy conditions (22), (23), and (30). This can be done as before, with the aid of a relation of the form (32).

§ 8. Forms of proof of the same type, but still more general, might be given by choosing M as well as $k$ to be some mean lying *between* the arithmetic and the geometric mean of the quantities with reference to which it is defined.

# Note on Tortuous Curves.

By JOHN MILLER, M.A.

The condition that the principal normals of one curve may also be the principal normals of a second curve is, as found by Bertrand, that a linear relation with constant coefficients should exist between the curvature and torsion of each curve. In seeking for pairs of curves such that the tangents, principal normals or binormals of one may be the tangents, principal normals or binormals of the other, there are six cases to be considered. The curves of Bertrand are furnished by one case, and a second case, that of evolutes and involutes, is also discussed in the text-books. Of the remaining four only one gives results worthy of mention. Bertrand's problem suggests the inquiry into the nature of the pair of curves when the binormal of one is the principal normal of the other. A certain quadratic relation of a simple character found to exist between the curvature and torsion of the second curve led me to a paper in the *Comptes Rendus* of 1893, by Demoulin, in which the problem had been generalised. His method of solution is different, and no explicit results as to the nature of the curves are given in the paper. Since no indication of the discussion of the problem is given in the text-books I have seen, I venture to submit a note of some results.

Let $(x, y, z)$ be a point on the curve;

$$\alpha, \beta, \gamma; \quad l, m, n; \quad \lambda, \mu, \nu;$$

the direction cosines of the tangent, principal normal and binormal at $(x, y, z)$; we shall take these lines to be so directed that by displacement they may be brought to coincide with the positive $x$, $y$ and $z$ axes respectively. Also let $1/R$ and $1/T$ be the curvature and torsion.

If $(\xi, \eta, \zeta)$ be the point on the second curve corresponding to $(x, y, z)$ then

$$\xi = x + a\lambda,$$
$$\eta = y + a\mu,$$
$$\zeta = z + a\nu,$$

where $a$ carrying its own sign is the distance between the points.

Taking the differentials and putting $\lambda d\xi + \mu d\eta + \nu d\zeta = 0$ we find that $da = 0$, that is, $a$ is a constant.

From Frenet's formulæ

$$\frac{da}{ds} = \frac{l}{R}, \quad \frac{dl}{ds} = -\frac{a}{R} - \frac{\lambda}{T}, \quad \frac{d\lambda}{ds} = \frac{l}{T},$$

we have

$$d\xi = \left(a + \frac{al}{T}\right)ds,$$
$$d\eta = \left(\beta + \frac{am}{T}\right)ds,$$
$$d\zeta = \left(\gamma + \frac{an}{T}\right)ds.$$

Denote corresponding quantities for the $(\xi, \eta, \zeta)$ curve by the same letters with suffixes.

Then

$$a_1 = \kappa\left(a + \frac{al}{T}\right),$$
$$\beta_1 = \kappa\left(\beta + \frac{am}{T}\right), \quad - \quad - \quad - (1)$$
$$\gamma_1 = \kappa\left(\gamma + \frac{an}{T}\right)$$

where $\kappa = \dfrac{ds}{ds_1}$.

By squaring and adding,

$$\kappa^2 = \frac{1}{1 + \dfrac{a^2}{T^2}}.$$

$$\therefore \quad ds_1 = \sqrt{\left(1 + \frac{a^2}{T^2}\right)}\, ds,$$

the positive sign of the root being taken so that $s$ and $s_1$ increase together.

Also $$aa_1 + \beta\beta_1 + \gamma\gamma_1 = \kappa = \cos\theta$$

where $\theta$ is the angle between the corresponding tangents.

It is seen from the formula $\frac{d\mu}{ds} = \frac{m}{T}$ that the torsion is positive when the positive binormal rotates round the tangent in the direction towards the centre of curvature. Hence the positive direction of $\theta$ being that from the tangent to the principal normal, we have

$$\cos\theta = \frac{1}{\sqrt{\left(1 + \frac{a^2}{T^2}\right)}}, \quad \tan\theta = \frac{a}{T}. \quad - \quad - \quad - \quad (2)$$

Differentiation of the three formulæ (1) gives

$$\frac{l_1}{R_1} ds_1 = d\kappa\left(a + \frac{al}{T}\right) + \kappa\left(\frac{l}{R} - \frac{aa}{RT} - \frac{a\lambda}{T^2} - \frac{al}{T^2}\frac{dT}{ds}\right) ds, \quad (3)$$

with two corresponding results.

Now let the binormal of the first curve be the principal normal of the second so that

$$\lambda = \epsilon l_1, \ \mu = \epsilon m_1, \ \nu = \epsilon n_1 \text{ where } \epsilon = \pm 1.$$

By multiplying the three equations (3) by $\lambda$, $\mu$, $\nu$ and adding we have

$$\frac{\epsilon ds_1}{R_1} = -\frac{a\kappa}{T^2} ds,$$

$$\text{or} \quad \frac{\epsilon}{R_1} = -\frac{a}{T^2 + a^2}. \quad - \quad - \quad - \quad - \quad (4)$$

Since the curvature is positive in Frenet's formulæ, $\epsilon = -1$ if $a$ is positive and $\epsilon = 1$ if $a$ is negative.

The squaring and addition of (3) gives, after inserting the value of $\kappa$,

$$\frac{1}{R_1^2} = \frac{a^2 T^2}{(T^2 + a^2)^3}\left(\frac{dT}{ds}\right)^2 - \frac{2aT^2}{R(T^2 + a^2)^2}\frac{dT}{ds} + \frac{T^2}{R^2(T^2 + a^2)} + \frac{a^2}{(T^2 + a^2)^2}$$

$$= \frac{a^2}{(T^2 + a^2)^2}.$$

$$\therefore \quad a^2 R^2\left(\frac{dT}{ds}\right)^2 - 2aR(T^2 + a^2)\frac{dT}{ds} + (T^2 + a^2)^2 = 0$$

$$\text{or} \quad aR\frac{dT}{ds} = T^2 + a^2.$$

$$\therefore \quad \tan^{-1}\left(\frac{T}{a}\right) = \int \frac{ds}{R} + \text{constant}.$$

Since $a$ occurs in one of the intrinsic equations of the curve, there can only be *one* curve of the second kind associated with it.

If $T = T_0$ when $s = 0$ and $S = \int_0^s \frac{ds}{R}$,

then $$T = \frac{aT_0 + a^2\tan S}{a - T_0\tan S}. \quad - \quad - \quad - \quad (5)$$

Let $s = f(R)$; then $S = \int_{R_0}^{R} \frac{f'(R)}{R} dR$,
where $R = R_0$ when $s = 0$.

A particular curve having this intrinsic equation free from the trigonometric function is got by putting

$$f'(R) = \frac{cR}{c^2 + R^2} \text{ or } s = \tfrac{1}{2}c\log\left(\frac{c^2 + R^2}{c^2 + R_0^2}\right);$$

$$\tan S \text{ is then } = \frac{c(R - R_0)}{c^2 + RR_0},$$

$$\text{and} \quad T = a\frac{R(R_0T_0 + ac) - c(aR_0 - cT_0)}{R(aR_0 - cT_0) + c(R_0T_0 + ac)}.$$

Let $\psi$ be the angle between the tangents at $s = 0$ and $s = s$ when the surface formed by the tangents is developed on a plane;

then $$\int_0 \frac{ds}{R} = \psi \text{ and } T = a\tan\psi.$$

Since from (2) $$\tan\theta = \frac{a}{T},$$

$$\tan\theta = \cot\psi.$$

$$\therefore \quad \psi = (2n + 1)\frac{\pi}{2} - \theta.$$

It remains to determine $T_1$;

$$\lambda_1 = \beta_1 n_1 - \gamma_1 m_1 = \epsilon(\beta_1 \nu - \gamma_1 \mu).$$

$$\therefore \quad \frac{d\lambda_1}{ds_1} \cdot \frac{ds_1}{ds} = \frac{\epsilon}{T}(\beta_1 n - \gamma_1 m).$$

$$\therefore \quad \frac{d\lambda_1}{ds_1} = \frac{\kappa^2\epsilon}{T}(n\beta - m\gamma) \text{ from (1)}.$$

But $\frac{d\lambda_1}{ds_1} = \frac{l_1}{T_1}$, $n\beta - m\gamma = \lambda$ and $l_1 = \epsilon\lambda$.

Hence $$\frac{1}{T_1}=\frac{T}{T^2+a^2} \qquad - \quad - \quad - \quad - \quad (6)$$

and, from (4), $TT_1 = -a\epsilon R_1$.

Also $$\frac{1}{T_1^2}+\frac{1}{R_1^2}=-\frac{\epsilon}{aR_1} \qquad - \quad - \quad - \quad (7)$$

where $\epsilon$ and $a$ are of opposite signs.

Referred to the tangent, principal normal and binormal at a point on a curve as axes, the equations of the axis of the osculating helix which has the same torsion as the curve at the point are

$$\frac{z}{x}=-\frac{T}{R},\ y=\frac{RT^2}{T^2+R^2}.$$

Here $$\frac{R_1T_1^2}{R_1^2+T_1^2}=-a\epsilon.$$

Also $$\frac{z}{x}=-\frac{T_1}{R_1}=\frac{a\epsilon}{T} \text{ from (6).}$$

But $\frac{z}{x}$ is the tangent of the angle this axis makes with the tangent to the curve, the angle being measured from the tangent to the binormal. Hence from the last two results and from (2), the tangent to the first curve is the axis of the helix which osculates the second curve and has the same torsion.

## The Conditions for the Reality of the Roots of an $n$-ic.

By Dr W. PEDDIE.

The homogeneous real linear transformation in $n$ variables such that, when these variables are used as a set of mutual rectangular coordinates, an $n$-dimensional sphere is transformed in an $n$-dimensional ellipsoid; $n$ mutually rectangular radii of t sphere become the $n$, mutually rectangular, principal radii of t ellipsoid. When these principal radii have not been rotated fr their original directions, the transformation is said to be *pure*, *irrotational*. Since these radii are necessarily real, the roots the $n$-ic for the determination of the $n$ principal elongations a necessarily real. If, therefore, we find the conditions which mu subsist amongst the $n^2$ constants which define the transformation, order that it may be pure, we have got conditions which are *sufficie* to ensure that the roots of the $n$-ic for the determination of t elongations along non-rotated lines shall be real.

Tait (*Proc. R.S.E.*, 1896, *Scientific Papers*, CXX.) first show that these conditions are not necessary, and proved that a sufficie condition, in three dimensions, is that the transformation shall decomposable into two superposed pure transformations. The sta ment is true generally. He gave the single relation which mu subsist amongst the nine constants in order that there should three real non-rotated lines, which are not in general mutua rectangular. He explicitly confined his investigation to transf mations capable of representing strain possible in ordinary matt but he remarked that it was easy to remove that restriction.

Muir (*Phil. Mag.*, 1896, Vol. XLIII.), not noticing Tai restriction, added explicitly to Tait's single condition other con tions regarding signs which are implicitly involved in the restrictio He applied Tait's process to the $n$-dimensional transformation, givi $(n-1)(n-2)/2$ equations of the same form as Tait's, and a rule

which they could be readily written down. In the investigation given below, $(n-1)(n-2)/2$ equations of a different type are found, and a compact general statement, which includes them all, is given. Muir's equations, along with other necessary but not independent relations, can of course be deduced from them. It is found also that Muir's conditions regarding signs are necessary if we remove the analogue of Tait's restriction, which is their equivalent.

The well-known condition that the transformation shall be pure is that the determinantal equation shall be axi-symmetrical. This is proved readily by considering the $n$ coordinates of a point, referred to $n$ rectangular axes, in $n$-dimensional space, say $x_1, \ldots, x_n$. Assume another set of such axes, with coordinates $\xi_1, \ldots, \xi_n$, as the set of non-rotated lines. Under these conditions $\xi_p$ becomes $\xi_p' = e_p \xi_p$, where $e_p$ is a constant. Let $l_{pq}$ denote the cosine of the angle between the axes of $\xi_p$ and $x_q$. Referring the point $(x_1, \ldots, x_n)$ to the $\xi$ axes, imposing the transformation which changes $\xi_p$ to $\xi_p'$, etc., and referring back to the $x$ axes, we get

$$x_r' = \sum_{p=1}^{p=n} . \, e_p l_{pr} \sum_{q=1}^{q=n} l_{pq} x_q,$$

$$x_q' = \sum_{p=1}^{p=n} . \, e_p l_{pq} \sum_{r=1}^{r=n} l_{pr} x_r,$$

so that the coefficient of $x_q$ in $x_r'$ is equal to the coefficient of $x_r$ in $x_q'$.

In accordance with Tait's method, we now seek to obtain the conditions which must hold amongst the coefficients, when the equation is not axi-symmetrical, in order that it shall be capable of being changed into an axi-symmetrical equation by a process which does not alter the roots.

Let the coefficient of $x_q$ in $x_r'$ be $c_{rq}$ while the coefficient of $x_r$ in $x_q'$ is $c_{qr}$. Tait's process consists in multiplying the $r$th row by $a_r$, and dividing the $q$th column by $a_q$, and so on. The coefficients now become

$$c_{rq}a_r/a_q, \quad c_{(r+1)q}a_{r+1}/a_q, \quad c_{r(q+1)}a_r/a_{q+1}, \quad c_{(r+1)(q+1)}a_{r+1}/a_{q+1},$$

with the corresponding set in which $q$ and $r$ are interchanged.

Hence, postulating axi-symmetry, and eliminating the $a$'s, we find

$$c_{qr}\,c_{(q+1)(r+1)}\,c_{r(q+1)}\,c_{(r+1)q}=c_{rq}\,c_{(r+1)(q+1)}\,c_{q(r+1)}\,c_{(q+1)r}.$$

The $(n-1)(n-2)/2$ equations of this type are conditions under which the roots shall be real.

If we use the term "image minors" with reference to any minor $c_{qr}\,c_{(q+1)(r+1)}-c_{q(r+1)}\,c_{(q+1)r}$ and the corresponding one in which $q$ and $r$ are interchanged, and if we use the term "cross products" with reference to the two quadruple products, we can say that

*The roots of an* n-ic *are real if the cross products of any pair of image minors, in the determinantal form of the equation, are equal.*

When $q+1=r$, the coefficient $c_{rr}$ is common to both cross products, so that the above condition includes Tait's condition for the case $n=3$. When $r=q$, the condition becomes a mere identity.

From these conditions it is easy to prove Muir's conditions, such as $c_{12}c_{24}c_{41}=c_{21}c_{14}c_{42}$; as also more complicated relations, such as

$$c_{n(n-1)}\,c_{(n-2)(n-3)}\ldots c_{21}c_{1n}=c_{n1}c_{12}\ldots c_{(n-2)(n-1)}c_{(n-1)n}.$$

Since we have $c_{pq}a_p{}^2=c_{qp}a_q{}^2$, we see that, so far as this condition goes, $c_{pq}$ and $c_{qp}$ might be of opposite signs if we regard $a_p$ *or* $a_q$ as an imaginary quantity. But, since $c_{pq}a_p/a_q$ and $c_{qp}a_q/a_p$ must be equal and real, all the $a$'s must be imaginary if one $a$ is so. Therefore $c_{pq}$ and $c_{qp}$ must have like signs when we consider real coefficients only. In the case in which the roots represent, in proper units, the squared frequencies of the fundamental vibrations of a system of $n$ masses, under the action of forces which are homogeneous and linear in the coordinates, $c_{pq}$ and $c_{qp}$ are necessarily of like sign since the masses are positive and the third law of motion holds.

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## On a simple theodolite suitable for use in schools.

By Loudon Arneil, M.A.

*Sixth Meeting, 11th May 1906.*

D. C. M'INTOSH, Esq., M.A., President, in the Chair.

## A proof that the middle points of parallel chords of a conic lie on a fixed straight line.

By Professor JACK.

FIGURES 12, 13, 14.

Let S be the focus of the conic, FX the directrix, $e$ the eccentricity.

Let V be the middle point of PP′.

Draw VK perpendicular to the directrix, and with centre V describe a circle, radius equal to $e$.VK.

Join SP, SP′ and draw radii V$p'$, V$p$, parallel to SP′, SP, and let PP′ meet the directrix in L. Then $p$, $p'$ are on the line SL.

Now $$\mathrm{PV} = \mathrm{VP'} \quad \text{(by hypothesis)}$$

$$\therefore \quad \frac{\mathrm{PV}}{\mathrm{VL}} = \frac{\mathrm{P'V}}{\mathrm{VL}},$$

$$\text{and} \quad \frac{\mathrm{PV}}{\mathrm{VL}} = \frac{\mathrm{S}p}{p\mathrm{L}} \quad \text{and} \quad \frac{\mathrm{P'V}}{\mathrm{VL}} = \frac{\mathrm{S}p'}{p'\mathrm{L}};$$

$$\therefore \quad \frac{\mathrm{S}p}{p\mathrm{L}} = \frac{\mathrm{S}p'}{p'\mathrm{L}};$$

$\therefore$ L$p'$S$p$ is a harmonic range,

and S is on the polar of L with regard to the circle.

Draw this polar, meeting PP′ in Y, the directrix in F and VK in H.

Now in the quadrilateral LYHK,

$$\angle \mathrm{LKH} = \angle \mathrm{LYH} = \frac{\pi}{2};$$

$\therefore$ the figure is cyclic.

$$\therefore \quad VH . VK = VY . VL = (e . VK)^2.$$

$$\therefore \quad VH = e^2 VK.$$

Now produce VF to meet SX (which is perpendicular to the directrix) in C;

$$\therefore \quad \frac{CS}{CX} = \frac{VH}{VK} = e^2;$$

$\therefore$ C is a fixed point; and so is F; and $\therefore$ V is always on the fixed line CF.

This proof applies to central conics (Fig. 12 has been drawn for the ellipse, Figs. 13 and 14 for the hyperbola).

The case of the parabola may be got by observing that C moves off to infinity and therefore all diameters are parallel to the axis; or it may be investigated as follows.

FIGURE 15.

Describe a circle with centre V and radius VK ($e$ being 1).

Join SP, SP′ and draw $Vp$, $Vp'$ parallel to them. Then as before $p$, $p'$ are on LS and $LpSp'$ is a harmonic range.

$\therefore$ S is on the polar of L with regard to the circle; (and this polar always goes through K)

$\therefore$ KV is perpendicular to the directrix, *i.e.*, parallel to the axis.

*The Converse* (for Central Conics).

With almost the same construction, we get

$$\frac{VH}{VK} = \frac{CS}{CX} = e^2, \quad \therefore \; VH = e^2 . VK.$$

$$\therefore \quad VH . VK = (e . VK)^2;$$

$$\therefore \quad VY . VL = (\text{the radius})^2;$$

$\therefore$ SF is polar of L;

$\therefore$ $LpSp'$ is a harmonic range;

$\therefore$ VP = VP′.

## Quantitative proofs of certain Algebraic Inequalities.

By JOHN DOUGALL, M.A.

**1.** By a quantitative proof of an inequality I mean one which exhibits the difference between the two magnitudes compared in a form which shows at a glance whether the difference is positive or negative. Such a proof not merely establishes the existence of the inequality, but also gives a measure of its amount.

The formula $\quad a^2+b^2-2ab=(a-b)^2,$

as a proof that $a^2+b^2>2ab$, is a characteristic example.

Another is the proof of the important theorem that when $p$ is a positive integer, and $x$ any positive number,

$$\frac{x^{p+1}-1}{p+1}>\frac{x^p-1}{p}$$

which is contained in the obvious identity

$$\begin{aligned}&p(x^{p+1}-1)-(p+1)(x^p-1)\\&=(x-1)^2(px^{p-1}+\overline{p-1}\,x^{p-2}+\ldots+1).\end{aligned}$$

Dr Muirhead, in two papers in Vols. XIX. and XXI. of these *Proceedings*, has made some interesting applications of the method; in the present communication I give a few additional developments suggested by a perusal of these papers.

The inequalities with which the paper deals admit of very simple *qualitative* proofs depending on the Theory of Equations, and they have been discussed from that point of view by Euler, Fort, and Schlömilch in the manner indicated in next article.

**2.** Taking $n$ real positive numbers $a_1$, $a_2$, ..., $a_n$, form the product $\quad (x+a_1y)(x+a_2y)\ldots\ldots(x+a_ny)$

and write its expansion in the form

$$A_0x^n + nA_1x^{n-1}y + \frac{n(n-1)}{1.2}A_2x^{n-2}y^2 + \dots + nA_{n-1}xy^{n-1} + A_ny^n.$$

If we differentiate $n - r - 1$ times as to $x$, and $r - 1$ times as to $y$, then by a fundamental theorem the resulting quadratic function, which to a constant factor is

$$A_{r-1}x^2 + 2A_rxy + A_{r+1}y^2$$

has its linear factors real.

Hence $A_r^2 - A_{r-1}A_{r+1}$ is positive. - - - - - - (a)

We have therefore

$$\frac{A_1}{A_0} > \frac{A_2}{A_1} > \frac{A_3}{A_2} > \dots > \frac{A_n}{A_{n-1}}.$$

In particular, if $s \nless r$

$$\frac{A_r}{A_{r-1}} > \frac{A_{s+1}}{A_s}$$

and $A_rA_s - A_{r-1}A_{s+1}$ is positive. - - - - - - (b)

Also $$\frac{A_r}{A_{r-1}} \cdot \frac{A_{r-1}}{A_{r-2}} \dots \frac{A_{r-p+1}}{A_{r-p}} > \frac{A_{s+1}}{A_s}\,\frac{A_{s+2}}{A_{s+1}} \dots \frac{A_{s+p}}{A_{s+p-1}},$$

that is $$\frac{A_r}{A_{r-p}} > \frac{A_{s+p}}{A_s}$$

or $A_rA_s - A_{r-p}A_{s+p}$ is positive. - - - - - - (c)

Again from (a) we have $A_r^{2r} > A_{r-1}^r\,A_{r+1}^r$, and therefore

$$\frac{A_r^{r+1}}{A_{r+1}^r} > \frac{A_{r-1}^r}{A_r^{r-1}} > \frac{A_{r-2}^{r-1}}{A_{r-1}^{r-2}} > \dots\dots > \frac{A_1^2}{A_2} > 1$$

and $A_r^{r+1} - A_{r+1}^r$ is positive. - - - - - - (d)

Generally, if we raise the inequalities

$$A_1^2 > A_0A_2$$
$$A_2^2 > A_1A_3$$
$$A_3^2 > A_2A_4$$
$$\dots\dots\dots\dots$$

to the positive integral powers $a_1$, $a_2$, $a_3$, ... and multiply, we get

$$A_1^{2a_1}A_2^{2a_2}A_3^{2a_3}\ldots\ldots > A_0^{a_1}A_1^{a_2}A_2^{a_1+a_3}A_3^{a_2+a_4}\ldots$$

or say, when the common factors are removed

$$A_r^{\alpha}A_s^{\beta}\ldots\ldots > A_p^{\gamma}A_q^{\delta}\ldots\ldots$$

so that $\qquad A_r^{\alpha}A_s^{\beta}\ldots\ldots - A_p^{\gamma}A_q^{\delta}\ldots\ldots$ is positive. - - ($e$)

Dr Muirhead, in Vol. XXI., proves ($a$) by expressing $A_r^2 - A_{r-1}A_{r+1}$ as a sum of terms each of which is manifestly positive. The problem proposed here is to do the same thing for the general form ($e$) and in particular for the special cases ($b$), ($c$), ($d$). Moreover we stipulate that the functions as the sum of which the difference ($e$) is expressed, besides being patently positive, shall be *integral* functions of the $a$'s. This restriction aside, the problem would be extremely simple. In fact, given a number of inequalities between positive quantities, such as

$$x_1 > y_1$$
$$x_2 > y_2$$
$$x_3 > y_3$$
$$\ldots\ldots\ldots$$

from which we can deduce $x_1x_2x_3\ldots > y_1y_2y_3\ldots$, nothing is easier than to write the difference

$$x_1x_2x_3\ldots\ldots - y_1y_2y_3\ldots\ldots$$

in such a form as to show its essentially positive character.

For example, we may write

$$x_1 = y_1 + (x_1 - y_1)$$
$$x_2 = y_2 + (x_2 - y_2), \text{ etc.,}$$

and then the extended product of the binomial factors

$$(y_1 + \overline{x_1 - y_1})(y_2 + \overline{x_2 - y_2})\ldots\ldots$$

contains $y_1y_2\ldots$ together with obviously positive terms.

Or better, we may write

$$x_1x_2 \quad - y_1y_2 \quad = x_2(x_1 - y_1) + y_1(x_2 - y_2)$$

$$x_1x_2x_3 - y_1y_2y_3 = (x_1x_2 - y_1y_2)x_3 + y_1y_2(x_3 - y_3)$$

$$= x_2x_3(x_1 - y_1) + y_1x_3(x_2 - y_2) + y_1y_2(x_3 - y_3)$$

the general result corresponding to which is obvious.

The form (*e*) *before common factors have been removed* from the two terms, could be treated in this way and expressed in terms of the differences (*a*). But the simplified form of (*e*) and in particular the forms (*b*), (*c*), (*d*) would thus be given as a sum of terms which, while plainly positive, would in general involve powers of the A's in the denominators,

$$e.g., \quad A_1A_2 - A_0A_3 = \frac{A_1}{A_2}(A_2^2 - A_1A_3) + \frac{A_3}{A_2}(A_1^2 - A_0A_2).$$

It will be shown, however, that by making use in the way thus suggested of forms of the type (*b*) in addition to those of the type (*a*) the difficulty about fractions can be surmounted.

What we do, then, is this:—*First*, we investigate expressions for $A_rA_s - A_{r-1}A_{s+1}$ where $s \nless r$, as a sum of evidently positive terms. The method used is different from Muirhead's, but some of the expressions found reduce to those given by him when $r = s$. *Second*, we express the differences (*c*), (*d*), (*e*) as linear functions of forms of the types (*a*) and (*b*), the coefficients of these functions being products of integral powers of the A's, or sums of such.

3. Given $n$ letters $a$, $b$, $c$, ..., which in the applications to inequalities we shall suppose to denote positive numbers;

$$\text{let} \quad P_r \equiv \Sigma abc \ldots\ldots \text{ to } r \text{ factors};$$

also let $P_0 = 1$, and when the integer $r$ is negative or greater than $n$, $P_r = 0$.

In $P_r$ let A = sum of the terms containing $a$,

B = „ „ „ „ „ $b$, etc.

In $P_s$ let A′ = sum of the terms *not* containing $a$,

B′ = „ „ „ „ „ „ $b$, etc.

Then we have

$$A + B + C + \ldots\ldots = rP_r,$$
$$A' + B' + C' + \ldots\ldots = (n-s)P_s,$$
$$\text{and}\quad \frac{A}{a} + \frac{B}{b} + \frac{C}{c} + \ldots\ldots = (n-r+1)P_{r-1};$$
$$aA' + bB' + cC' + \ldots\ldots = (s+1)P_{s+1}$$

as we see at once by counting the number of times any particular term occurs in the expressions on the left.

Thus $r(n-s)P_rP_s - (n-r+1)(s+1)P_{r-1}P_{s+1}$

$$= (A + B + \ldots)(A' + B' + \ldots) - \left(\frac{A}{a} + \frac{B}{b} + \ldots\right)(aA' + bB' + \ldots)$$
$$= \Sigma\{AB'(1 - b/a) + A'B(1 - a/b)\}$$
$$= \Sigma(a-b)\left(\frac{A}{a}B' - \frac{B}{b}A'\right). \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad (1)$$

Now we may write

$$P_r = abT + (a+b)S + R,$$
$$P_s = abT' + (a+b)S' + R'$$

where $T$, $S$, $R$, $T'$, $S'$, $R'$ do not contain $a$ or $b$.

With this notation $A = a(bT + S)$,

$$B = b(aT + S)$$

and $A' = bS' + R'$,

$$B' = aS' + R'.$$

The typical term on the right of (1) is then

$$(a-b)\{(bT + S)(aS' + R') - (aT + S)(bS' + R')\}$$
$$= (a-b)^2(SS' - TR')$$
$$= (a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab} - P_{r-2}^{ab}P_s^{ab}) \quad - \quad - \quad - \quad - \quad - \quad - \quad (2)$$

where we use the symbol $P_t^{ab}$ to denote $\Sigma cd \ldots$ to $t$ factors taken from the $n-2$ letters left when $a$, $b$ are excluded from the original $n$.

Hence from (1)

$$r(n-s)P_rP_s - (n-r+1)(s+1)P_{r-1}P_{s+1}$$
$$= \Sigma(a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab} - P_{r-2}^{ab}P_s^{ab}). \quad - \quad - \quad - \quad - \quad - \quad (3)$$

On this formula the whole of the succeeding work is based.

4. If the cofactor of $(a-b)^2$ in (2) be fully expanded, the numerical coefficient of every term will be positive, provided $s$ is not less than $r$, as we shall always suppose to be the case.
For consider $P_rP_s - P_{r-1}P_{s+1}$, involving $t$ letters $a_1, a_2, \ldots, a_t$.

The product $P_rP_s$, when the number of letters is sufficient, involves terms of the types

$$a_1a_2 \ldots\ldots a_r \,.\, a_{r+1} \ldots a_{r+s}\quad , \text{ coefficient} = (r+s)\,!/(r\,!\,s\,!)$$

$$a_1^2a_2 \ldots\ldots a_r \,.\, a_{r+1} \ldots a_{r+s-1}, \text{ coefficient} = (r+s-2)\,!/(r-1)\,!\,(s-1)\,!$$

$$a_1^2a_2^2a_3 \ldots a_r \,.\, a_{r+1} \ldots a_{r+s-2}, \text{ coefficient} = (r+s-4)\,!/(r-2)\,!\,(s-2)\,!$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_1^2a_2^2 \ldots\ldots a_r^2 \,.\, a_{r+1} \ldots a_s\quad , \text{ coefficient} = 1.$$

When $r+s>t$, some of the earlier terms will not occur, but those terms which do occur have the coefficient stated.

$P_{r-1}\,P_{s+1}$ will involve terms of the same types, except the last, and the coefficients will be

$$(r+s)\,!/(r-1)\,!\,(s+1)\,!,\quad (r+s-2)\,!/(r-2)\,!\,s\,!,\quad \text{etc.}$$

In $P_rP_s - P_{r-1}\,P_{s+1}$ the coefficient of the terms containing $p$ squares will therefore be

$$(r+s-2p)\,!\,\{1/(r-p)\,!\,(s-p)\,! - 1/(r-p-1)\,!\,(s-p+1)\,!\}$$
$$= (s-r+1)(r+s-2p)\,!/(r-p)\,!\,(s-p+1)\,!$$

which is positive.

We may put this result in the form

$$P_rP_s - P_{r-1}\,P_{s+1}$$
$$= \Sigma a_1^2a_2^2 \ldots a_r^2\, a_{r+1} \ldots a_s$$
$$+ (s-r+1)\Sigma a_1^2a_2^2 \ldots a_{r-1}^2 a_r \ldots a_{s+1} + \ldots$$
$$+ (s-r+1)\frac{(r+s-2p)\,!}{(r-p)\,!\,(s-p+1)\,!}\Sigma a_1^2a_2^2 \ldots\ldots a_p^2 a_{p+1} a_{p+2} \ldots\ldots a_{r+s-p} + \ldots\ldots \quad (4)$$

When $r+s \not> t$, the last term will be

$$\frac{(s-r+1)(r+s)\,!}{r\,!\,(s+1)\,!}\Sigma a_1a_2 \ldots a_{r+s},$$

but when $r+s>t$, the series will stop when $r+s-p=t$, that is for $p=r+s-t$. If we substitute from (4) in the right of (3) we obtain a formula for the left of (3) as a sum of positive terms. For the case $r=s$ this is Muirhead's formula (33), Vol. XXI., page 156.

5. From the equation (3) itself we can derive other interesting expansions of $P_rP_s - P_{r-1}P_{s+1}$ serving like (4) to show that this function contains only positive terms.

Consider first the form which (3) takes when in addition to the $n$ letters $a, b, c, \ldots$, there are other $\nu$ letters $\alpha, \beta, \gamma, \ldots$. This may be written

$$\begin{aligned} & r(n+\nu-s)P_rP_s - (n+\nu-r+1)(s+1)\, P_{r-1}P_{s+1} \\ = \; & \Sigma(a-b)^2(P^{ab}_{r-1}\, P^{ab}_{s-1} - P^{ab}_{r-2}\, P^{ab}_{s}) \\ & + \Sigma(a-\alpha)^2(P^{a\alpha}_{r-1}\, P^{a\alpha}_{s-1} - P^{a\alpha}_{r-2}\, P^{a\alpha}_{s}) \\ & + \Sigma(\alpha-\beta)^2(P^{\alpha\beta}_{r-1}\, P^{\alpha\beta}_{s-1} - P^{\alpha\beta}_{r-2}\, P^{\alpha\beta}_{s}). \end{aligned}$$

Now put $\alpha, \beta, \gamma, \ldots$ all equal to 0, and this becomes

$$\begin{aligned} & r(n+\nu-s)P_rP_s - (n+\nu-r+1)(s+1)P_{r-1}P_{s+1} \\ = \; & \Sigma(a-b)^2(P^{ab}_{r-1}\, P^{ab}_{s-1} - P^{ab}_{r-2}\, P^{ab}_{s}) \\ & + \; \nu\Sigma a^2 \; (P^{a}_{r-1}\, P^{a}_{s-1} - P^{a}_{r-2}\, P^{a}_{s}) \end{aligned}$$

where, of course, only the $n$ letters $a, b, c, \ldots$ are now involved.

$\nu$ being arbitrary, we must have

$$rP_rP_s - (s+1)P_{r-1}\, P_{s+1} = \Sigma a^2(P^{a}_{r-1}\, P^{a}_{s-1} - P^{a}_{r-2}\, P^{a}_{s}) \quad \cdot \quad (5)$$

which may be written in either of the forms

$$(s+1)(P_rP_s - P_{r-1}P_{s+1}) = (s-r+1)P_r\;\; P_s \;\; + \Sigma a^2(P^{a}_{r-1}P^{a}_{s-1} - P^{a}_{r-2}P^{a}_{s}), \quad (6)$$

$$r(P_rP_s - P_{r-1}P_{s+1}) = (s-r+1)P_{r-1}P_{s+1} + \Sigma a^2(P^{a}_{r-1}P^{a}_{s-1} - P^{a}_{r-2}P^{a}_{s}). \quad (6)'$$

(6) or (6)′ may be used as a reduction formula for expressing $P_rP_s - P_{r-1}P_{s+1}$ as a sum of positive terms. Taking (6) we get

$$P^{a}_{r-1}\, P^{a}_{s-1} - P^{a}_{r-2}P^{a}_{s} = \frac{s-r+1}{s}\, P^{a}_{r-1}\, P^{a}_{s-1} + \frac{1}{s}\, \Sigma b^2(P^{ab}_{r-2}\, P^{ab}_{s-2} - P^{ab}_{r-3}\, P^{ab}_{s-1})$$

and therefore by substitution in (6)

$$P_rP_s - P_{r-1}P_{s+1} = \frac{s-r+1}{s+1}P_rP_s + \frac{s-r+1}{s.s+1}\Sigma a^2 P_{r-1}^{a} P_{s-1}^{a}$$
$$+\frac{2}{s.s+1}\Sigma a^2b^2(P_{r-2}^{ab}P_{s-2}^{ab} - P_{r-3}^{ab}P_{s-1}^{ab}).$$

We can apply (6) again to the last term, and so on, and obtain finally

$$P_rP_s - P_{r-1}P_{s+1} = (s-r+1)\frac{1}{s+1}P_rP_s + (s-r+1)\frac{1}{s+1.s}\Sigma a^2P_{r-1}^{a}P_{s-1}^{a}$$
$$+(s-r+1)\frac{1.2}{s+1.s.s-1}\Sigma a^2b^2P_{r-2}^{ab}P_{s-2}^{ab} + \dots\dots$$
$$+\frac{r!}{s+1.s\dots\dots(s-r+2)}\Sigma(a^2b^2\dots,\ r \text{ letters})P_{s-r}^{ab\dots,\ r\text{ letters}}. \quad (7)$$

A similar formula can be derived from (6)′.

If we use the formula (7) to develop (3) we get

$$\frac{s}{s-r+1}\{r(n-s)P_rP_s - (n-r+1)(s+1)P_{r-1}P_{s+1}\}$$
$$= \Sigma(a-b)^2P_{r-1}^{ab}P_{s-1}^{ab} + \frac{1}{s-1}\Sigma(a-b)^2c^2P_{r-2}^{abc}P_{s-2}^{abc}$$
$$+\frac{1.2}{s-1.s-2}\Sigma(a-b)^2c^2d^2P_{r-3}^{abcd}P_{s-3}^{abcd} + \dots\dots$$
$$+\frac{1.2\dots(r-1)}{s-1.s-2\dots(s-r+1)}\Sigma(a-b)^2(c^2d^2\dots,\ \overline{r-1}\text{ letters})P_{s-r}^{ab\dots,\ (r+1)\text{ letters}} \quad (8)$$

which when $r = s$ becomes Muirhead's (38), page 157, Vol. XXI.

6. By a simple transformation we can derive from (8) another formula which is perhaps simpler and in certain cases will contain fewer terms.

One special case we may note at once. Put $s = n - 1$ in (3) and we have immediately

$$rP_rP_{n-1} - (n-r+1)nP_{r-1}P_n = \Sigma(a-b)^2P_{r-1}^{ab}P_{n-2}^{ab},$$

since $P_{n-1}^{ab}$, referring to only $n-2$ letters, is zero. For this case (8) is clearly a much longer formula.

For the general case, apply (8) not to $a, b, c, \dots$ themselves, but to their reciprocals $1/a$, $1/b$, $1/c$, ....

Obviously $P_r$ becomes $P_{n-r}/P_n$; $P_{r-1}^{ab}$ becomes $P_{n-r-1}^{ab}/P_{n-2}^{ab}$, that is, $ab\,P_{n-r-1}^{ab}/P_n$, and so on. Hence, multiplying every term by $P_n^2$,

$$\frac{s}{s-r+1}\{r(n-s)P_{n-r}P_{n-s}-(n-r+1)(s+1)P_{n-r+1}P_{n-s-1}\}$$

$$= \Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2 a^2b^2 P_{n-r-1}^{ab}P_{n-s-1}^{ab}+\frac{1}{s-1}\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2\frac{1}{c^2}\,.\,a^2b^2c^2\,P_{n-r-1}^{abc}\,P_{n-s-1}^{abc}$$

$$+\frac{1\,.\,2}{s-1\,.\,s-2}\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2\frac{1}{c^2d^2}\,.\,a^2b^2c^2d^2\,P_{n-r-1}^{abcd}\,P_{n-s-1}^{abcd}+\ldots\ldots$$

$$+\frac{1\,.\,2\ldots(r-1)}{s-1\,.\,s-2\ldots(s-r+1)}\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2\frac{1}{c^2d^2\ldots,\,(r-1)\text{ letters}}(a^2b^2c^2d^2\ldots,\,\overline{r+1}\text{ letters})(P_{n-r-1}P_{n-s-1})^{abc\ldots,\,r+1\text{ letters}}$$

or, changing $r$ into $n-s$ and $s$ into $n-r$,

$$\frac{n-r}{s-r+1}\{r(n-s)P_rP_s-(n-r+1)(s+1)\,P_{r-1}\,P_{s+1}\}$$

$$= \Sigma(a-b)^2P_{r-1}^{ab}P_{s-1}^{ab}+\frac{1}{n-r-1}\Sigma(a-b)^2P_{r-1}^{abc}P_{s-1}^{abc}+\frac{1\,.\,2}{n-r-1\,.\,n-r-2}\Sigma(a-b)^2P_{r-1}^{abcd}P_{s-1}^{abcd}+\ldots\ldots$$

$$+\frac{1\,.\,2\ldots(n-s-1)}{n-r-1\,.\,n-r-2\ldots(s-r+1)}\Sigma(a-b)^2(P_{r-1}\,P_{s-1})^{abc\ldots,\,\overline{n-s+1}\text{ letters}}.$$

7. By a slightly different method of manipulating (3) we can obtain still another expression for $r(n-s)P_rP_s-(n-r+1)(s+1)P_{r-1}P_{s+1}$ as a sum of positive terms.

We can write (3) in the form

$$(n-r+1)(s+1)(P_rP_s-P_{r-1}P_{s+1})$$
$$=(n+1)(s-r+1)P_rP_s+\Sigma(a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab}-P_{r-2}^{ab}P_s^{ab})$$

and use this as a formula of reduction to expand the last term on the right.

We thus find

$$r(n-s)P_rP_s-(n-r+1)(s+1)P_{r-1}P_{s+1}$$
$$\{=\Sigma(a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab}-P_{r-2}^{ab}P_s^{ab})\}$$

$$=\frac{(n-1)(s-r+1)}{s(n-r)}\Sigma(a-b)^2P_{r-1}^{ab}P_{s-1}^{ab}+\frac{1.2.n-3.s-r+1}{s.s-1.n-r.n-r-1}\Sigma(a-b)^2(c-d)^2(P_{r-2}P_{s-2})^{abcd}$$

$$+\dots+\frac{q!(n+1-2q)(s-r+1)}{s.s-1..s-q+1.n-r.n-r-1..n-r-q+1}\Sigma(a-b)^2(c-d)^2.., 2q \text{ letters } (P_{r-q}P_{s-q})^{abcd\dots,\text{ these } 2q \text{ letters}}$$

$$+\dots\dots\dots\dots\dots\dots \qquad (10)$$

For the last term $q=$ the smaller of $r$, $n-s$.

The transformation of Art. 6 when applied to (10) simply reproduces the formula.

8. The preceding formulæ, viz., (8), (9), (10) and the result of substituting (4) in (3) furnish various expressions for the difference

$$r(n-s)P_rP_s-(n-r+1)(s+1)P_{r-1}P_{s+1}$$

as a sum of terms obviously positive when $a$, $b$, $c$, etc., are positive and not all equal, and $r$ is less than or equal to $s$.

If as in Art. 2 we put ${}_nC_r \,.\, A_r$ for $P_r$, and so on, this difference takes the form $\quad r(n-s)\,{}_nC_r \,.\, {}_nC_s(A_rA_s-A_{r-1}A_{s+1})$.

We have thus obtained expressions evidently positive for

$$A_r^2-A_{r-1}A_{r+1}$$

$$\text{and}\quad A_rA_s-A_{r-1}A_{s+1}\quad (r<s)$$

thus accounting for cases ($a$) and ($b$) of Art. 2.

In accordance with the statement at the end of that Article, the general problem now before us is to throw any difference of the type ($e$) into the form of a sum of terms such as $K(A_rA_s-A_{r-1}A_{s+1})$ where $r \not> s$ and K is a positive product of the powers of the A's; a problem of some interest even apart from the application to inequalities.

9. The forms ($c$) and ($d$) are particular cases of ($e$), but as they are specially simple and interesting, we shall deal with them here individually before proceeding to a method applicable to the general form.

We have immediately

$$\begin{aligned} A_rA_s-A_{r-p}A_{s+p} &= (A_rA_s-A_{r-1}A_{s+1})+(A_{r-1}A_{s+1}-A_{r-2}A_{s+2})+\ldots \\ &\quad +(A_{r-p+1}A_{s+p-1}-A_{r-p}A_{s+p}). \qquad (11) \end{aligned}$$

$$\begin{aligned} A_r^{r+1}-A_0A_{r+1}^r &= (A_r^{r+1}-A_{r-1}A_r^{r-1}A_{r+1})+(A_{r-1}A_r^{r-1}A_{r+1}-A_{r-2}A_r^{r-2}A_{r+1}^2) \\ &\quad +(A_{r-2}A_r^{r-2}A_{r+1}^2-A_{r-3}A_r^{r-3}A_{r+1}^3)+\ldots\ldots \\ &\quad +(A_2A_r^2A_{r+1}^{r-2}-A_1A_rA_{r+1}^{r-1})+(A_1A_rA_{r+1}^{r-1}-A_0A_{r+1}^r) \\ &= A_r^{r-1}(A_r^2-A_{r-1}A_{r+1})+A_r^{r-2}A_{r+1}(A_{r-1}A_r-A_{r-2}A_{r+1}) \\ &\quad +A_r^{r-3}A_{r+1}^2(A_{r-2}A_r-A_{r-3}A_{r+1})+\ldots\ldots \\ &\quad +A_rA_{r+1}^{r-2}(A_2A_r-A_1A_{r+1})+A_{r+1}^{r-1}(A_1A_r-A_0A_{r+1}). \qquad (12) \end{aligned}$$

10. The general difference is what is obtained from

$$A_1^{2a_1} A_2^{2a_2} .. A_t^{2a_t} - A_0^{a_1} A_1^{a_2} A_2^{a_1+a_3} A_3^{a_2+a_4} .. A_{t-1}^{a_{t-2}+a_t} A_t^{a_{t-1}} A_{t+1}^{a_t}, \quad (13)$$

when all factors, in the form of powers of A's, which are common to the two terms are removed.

Some of the $a$'s may not occur, that is, they may be zero ; but none of them are to be negative ; when this condition is fulfilled we shall call (13) or (13) simplified a *proper* difference.

Even after simplification, note that if $a_p$ and $a_t$ are the $a$'s of lowest and highest order, then $A_{p-1}$ and $A_{t+1}$ are the lowest and highest A's which occur, and they occur in the *negative* term in the form $A_{p-1}^{a_p}$, $A_{t+1}^{a_t}$.

Any difference being given, even in its simplified form, its $a$'s are determinate and can easily be found. Thus suppose that a given difference is

$$A_1^{x_1} A_2^{x_2} \ldots A_t^{x_t} - A_0^{y_0} A_1^{y_1} A_2^{y_2} \ldots A_t^{y_t} A_{t+1}^{y_{t+1}} \quad \cdot \quad (14)$$

where, if the difference has been simplified, one of $x_r$, $y_r$ is zero.

If this is derivable from (13) by mere elision of common factors, then

$$\left.\begin{array}{lll} \text{considering} & A_0, \text{ we get} & a_1 = y_0 \\ & A_1, & a_2 - 2a_1 = y_1 - x_1 \\ & A_2, & a_3 - 2a_2 + a_1 = y_2 - x_2 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ & A_{t-1}, & a_t - 2a_{t-1} + a_{t-2} = y_{t-1} - x_{t-1} \\ & A_t\ , & a_{t-1} - 2a_t = y_t - x_t \\ & A_{t+1}, & a_t = y_{t+1} \end{array}\right\} \cdot \quad \cdot \quad (15)$$

The first $t$ of these equations give in succession $a_1, a_2, \ldots, a_t$. The remaining two equations give

$$y_0 + (y_1 - x_1) + (y_2 - x_2) + \ldots + (y_t - x_t) + y_{t+1} = 0$$

and $\quad (y_1 - x_1) + 2(y_2 - x_2) + 3(y_3 - x_3) + \ldots + t(y_t - x_t) + (t+1)y_{t+1} = 0.$

These are the conditions that (14) should be derivable from a form of the type (13), and express the facts that the two terms of (14) are homogeneous in the A's and also in $a$, $b$, $c$, ....

The values of the $a$'s are easily found explicitly from (15). Thus

$$a_1 = y_0,$$
$$a_2 - a_1 = y_0 + (y_1 - x_1),$$
$$a_3 - a_2 = y_0 + (y_1 - x_1) + (y_2 - x_2), \text{ etc.}$$

so that

$$\left.\begin{aligned} a_1 &= y_0, \\ a_2 &= 2y_0 + (y_1 - x_1), \\ a_3 &= 3y_0 + 2(y_1 - x_1) + (y_2 - x_2), \\ a_4 &= 4y_0 + 3(y_1 - x_1) + 2(y_2 - x_2) + (y_3 - x_3), \text{ etc.} \end{aligned}\right\} \quad (16)$$

For example, for $A_rA_s - A_{r-p}A_{s+p}$ $(s \nless r)$
the $a$'s that occur are

$$\left.\begin{aligned} &a_{r-p+1} = 1,\ a_{r-p+2} = 2,\ a_{r-p+3} = 3,\ \ldots,\ a_r = p; \\ &a_{r+1} = a_{r+2} = \ldots = a_s = p; \\ &a_{s+1} = p - 1,\ a_{s+2} = p - 2,\ \ldots\ldots,\ a_{s+p-1} = 1. \end{aligned}\right\}$$

In particular, for $A_rA_s - A_{r-1}A_{s+1}$

$$a_r = a_{r+1} = a_{r+2} = \ldots = a_s = 1.$$

Again for $A_r^{r+1} - A_0A_{r+1}^r$

$$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ \ldots,\ a_r = r.$$

11. It is now easy to explain a method of reducing any given proper difference (14) to the shape we desire.

For brevity, denote the given difference by $X - Y$.

Then the process we will give is a step-by-step one, the first step being to put $X - Y$ in the form

$$(X - Y_1) + (Y_1 - Y)$$

where (i) $Y_1 - Y$ is the product of powers of A's by a proper difference of the elementary type $A_rA_s - A_{r-1}A_{s+1}$

and (ii) $X - Y_1$ is a proper difference with some of its $a$'s smaller, and none larger, than those of $X - Y$.

Suppose that $A_{r-1}A_{s+1}$ occurs in $Y$ as a factor.

For $Y_1$ take what $Y$ becomes when this factor is replaced by $A_rA_s$.

Then (i) $Y_1 - Y$ is clearly $(A_r A_s - A_{r-1} A_{s+1}) \times$ a product of powers of A's,

and (ii) $X - Y_1$ is a difference in which the values of all the $a$'s from $a_r$ to $a_s$ inclusive are reduced by 1 from their values in $X - Y$, the other $a$'s remaining unchanged; as may easily be verified from equations (16).

More generally, if $(A_{r-1}A_{s+1})^m$ occurs in Y as a factor, and we form $Y_1$ by replacing this by $(A_r A_s)^m$, then

(i) $Y_1 - Y$ will be $(A_r^m A_s^m - A_{r-1}^m A_{s+1}^m) \times$ a product of powers of A's, that is, $(A_r A_s - A_{r-1} A_{s+1}) \times$ an integral function of A's with positive coefficients;

and (ii) $X - Y_1$ will have all the $a$'s from $a_r$ to $a_s$ inclusive reduced by $m$ from their values in $X - Y$.

For the success of this process of reduction based on replacing $A_{r-1}A_{s+1}$ in Y by $A_r A_s$ it is necessary

(i) that $A_{r-1}A_{s+1}$ should occur in Y;

(ii) that $X - Y_1$ should still be a *proper* difference, which requires that none of the new $a$'s should be negative, or that in $X - Y$ all the $a$'s from $a_r$ to $a_s$ inclusive should occur.

Both these conditions will be fulfilled if we take for $r$ the index of the lowest $a$ that occurs in $X - Y$, in which case $A_{r-1}$ certainly occurs in Y, as pointed out near the beginning of Art. 10; and if at the same time we choose $s$ so that $a_{s+1}$ is the first $a$ after $a_r$ which does *not* occur in Y, in which case $A_{s+1}$ certainly occurs in Y, viz., raised to the power $a_s + a_{s+2}$, as we see from (13) since $A_{s+1}$ does not occur in X.

More generally, we can replace at one step $(A_{r-1}A_{s+1})^m$ by $(A_r A_s)^m$ provided, with $r$ and $s$ chosen as just explained, every $a$ from $a_r$ to $a_s$ inclusive is equal to $m$ at least.

Having thus got $X - Y = (X - Y_1) + (Y_1 - Y)$ we proceed in the same way with $X - Y_1$ and finally obtain

$$X - Y = (X - Y_n) + (Y_n - Y_{n-1}) + \ldots + (Y_2 - Y_1) + (Y_1 - Y)$$

where each of the differences in brackets is of the type

$$(A_r A_s - A_{r-1} A_{s+1}) \times$$

an integral function of A's with positive coefficients.

The reader may take the results of Art. 9 as easy examples of this process.

As another example consider the case

$$X = A_2^7 A_4^4 A_6^4 ; \quad Y = A_0 A_1^4 A_3^3 A_5^4 A_7^3 .$$

The scheme of $a$'s, originally, and after each step, is

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $X-Y$. | 1 | 6 | 4 | 5 | 2 | 3 |
| $X-Y_1$. | | 5 | 3 | 4 | 1 | 2 |
| $X-Y_2$. | | 4 | 2 | 3 | | 1 |
| $X-Y_3$. | | 2 | | 1 | | 1 |
| $X-Y_4$. | | | | 1 | | 1 |
| $X-Y_5$. | | | | | | 1 |

Therefore

$$Y_1 = A_1^5 A_3^3 A_5^4 A_6 A_7^2$$
$$Y_2 = A_1^4 A_2 A_3^3 A_5^4 A_6^2 A_7$$
$$Y_3 = A_1^2 A_2^3 A_3^3 A_4^2 A_5^2 A_6^2 A_7$$
$$Y_4 = A_2^7 A_3 A_4^2 A_5^2 A_6^2 A_7$$
$$Y_5 = A_2^7 A_4^4 A_5 A_6^3 A_7$$

and

$$\begin{aligned} X - Y = {} & (X - Y_5) + (Y_5 - Y_4) + (Y_4 - Y_3) + (Y_3 - Y_2) + (Y_2 - Y_1) + (Y_1 - Y) \\ = {} & A_2^7 A_4^4 A_6^3 (A_6^2 - A_5 A_7) \\ & + A_2^7 A_4^2 A_5 A_6^2 A_7 (A_4^2 - A_3 A_5) \\ & + A_2^3 A_3 A_4^2 A_5^2 A_6^2 A_7 (A_2^4 - A_1^2 A_3^2) \\ & + A_1^2 A_2 A_3^3 A_5^2 A_6^2 A_7 (A_2^2 A_4^2 - A_1^2 A_5^2) \\ & + A_1^4 A_3^3 A_5^4 A_6 A_7 (A_2 A_6 - A_1 A_7) \\ & + A_1^4 A_3^3 A_5^4 A_7^2 (A_1 A_6 - A_0 A_7), \end{aligned}$$

where we may replace

$A_2^4 - A_1^2 A_3^2$ by $(A_2^2 + A_1 A_3)(A_2^2 - A_1 A_3)$

and $A_2^2 A_4^2 - A_1^2 A_5^2$ by $(A_2 A_4 + A_1 A_5)(A_2 A_4 - A_1 A_5)$.

12. The preceding process for decomposing a "proper difference" is simple and straightforward, but it is not to be inferred that the decomposition obtained is the only one possible which would serve the purpose. Take for example

$$A_3^4 - A_0A_4^3.$$

As in Art. 9 this is

$$A_3^2(A_3^2 - A_2A_4) + A_3A_4(A_2A_3 - A_1A_4) + A_4^2(A_1A_3 - A_0A_4).$$

But it may also be put in the form

$$(A_3^2 + A_2A_4)(A_3^2 - A_2A_4) + A_4^2(A_2^2 - A_1A_3) + A_4^2(A_1A_3 - A_0A_4).$$

The latter form, however, contains a term more than the other, if we count $(A_rA_s - A_{r-1}A_{s+1})$ as one term, but $(A_r^mA_s^m - A_{r-1}^mA_{s+1}^m)$, *i.e.*, $(A_rA_s - A_{r-1}A_{s+1})(A_r^{m-1}A_s^{m-1} + \ldots)$ as $m$ terms.

So, generally, it may happen that it is possible to make two or more steps of the single step by which in the above process we change $Y_a$ into $Y_b$ by replacing $A_{r-1}A_{s+1}$ by $A_rA_s$. For if an A, say $A_{x+1}$, intermediate to $A_{r-1}$ and $A_{s+1}$ occur in $Y_a$, as it may very well do, we may first change $Y_a$ into $Y_c$, say, by replacing $A_{r-1}A_{x+1}$ by $A_rA_x$, and then change $Y_c$ into $Y_b$ by replacing $A_xA_{s+1}$ by $A_{x+1}A_s$. The only effect of this is to split up a single term of the original final result into two, and therefore to increase the total number of terms by one.

It would therefore seem that the decomposition we have given is that which leads to the minimum number of terms.

13. An inequality of perennial interest is that which holds between the Arithmetic and Geometric Means of a set of positive numbers, and we may conclude by examining what the preceding methods make of this important example.

The Arithmetic Mean of the $n$ numbers $a$, $b$, $c$, ... is

$$(a + b + c + \ldots)/n \text{ or } A_1;$$

their Geometric Mean is $(abc\ldots)^{1/n}$, *i.e.*, $A_n^{1/n}$.

We prove $A_1 > A_n^{1/n}$ by exhibiting $A_1^n - A_n$ in an explicitly positive form.

To look for a moment at a more general case, note that by ($d$)

$$A_r^{1/r} > A_{r+1}^{1/\overline{r+1}},$$

and therefore $\qquad A_r^{1/r} > A_s^{1/s}$ if $s > r$;

hence $\qquad A_r^s - A_s^r$ or $A_r^s - A_0^{s-r}A_s^r$ is positive,

is, in fact, a *proper* difference when $s > r$.

The $a$'s of this difference are by (16)

$$a_1 = s - r, \quad a_2 = 2(s-r), \quad a_3 = 3(s-r), \ldots, \quad a_r = r(s-r);$$

$$a_{r+1} = r(s-r-1), \quad a_{r+2} = r(s-r-2), \ldots\ldots \quad , \quad a_{s-1} = r.$$

If $r$ and $s$ are given numerically, the reduction by the method of Art. 11 is very simple; for, since the $a$'s diminish steadily from $a_r$ towards both ends of the series, it is clear that at every step we have merely to replace in the negative term the lowest A by the A one higher and the highest A by the A one lower. But it would be somewhat difficult to state a *general* formula for the result, the form of which in fact depends on the relative magnitudes of the various multiples of $r$ and $s$.

For the special case $A_1^n - A_0^{n-1}A_n$ the difficulty does not exist.

The $a$'s are $a_1 = n-1$, $a_2 = n-2$, $a_3 = n-3$, ..., $a_n = 1$;

and the formula, which may be verified at a glance without reference to the general method, is

$$A_1^n - A_n = A_1^{n-2}(A_1^2 - A_2) + A_1^{n-3}(A_1A_2 - A_3) + A_1^{n-4}(A_1A_3 - A_4) + \ldots$$
$$+ A_1(A_1A_{n-2} - A_{n-1}) + (A_1A_{n-1} - A_n).$$

Now (3) of Art. 3 gives

$$(n-s)P_1P_s - n(s+1)P_{s+1} = \Sigma(a-b)^2P_{s-1}^{ab}$$

*i.e.*, $$n^2\frac{(n-1)(n-2)\ldots(n-s)}{1.2\ldots\ldots s}(A_1A_s - A_{s+1}) = \Sigma(a-b)^2P_{s-1}^{ab}.$$

Hence

$$A_1^n - A_n = \frac{1}{n^2}\Sigma(a-b)^2\left\{\frac{1}{n-1}A_1^{n-2} + \frac{1.2}{n-1.n-2}A_1^{n-3}P_1^{ab}\right.$$
$$+\frac{1.2.3}{n-1.n-2.n-3}A_1^{n-4}P_2^{ab} + \ldots$$
$$\left. + \frac{1}{n-1}A_1P_{n-3}^{ab} + P_{n-2}^{ab}\right\}.$$

# Notes on the Apollonian Problem and the allied theory.

By John Dougall, M.A.

1. This paper contains a number of investigations, more or less connected, on the theory of systems of circles. In such a well-worn field one does not expect to have hit upon much that is absolutely new, but it may be hoped that there is sufficient freshness of treatment to give the paper some interest even where it deals with results already known.

The possibility of some of the elements of a figure being imaginary is contemplated throughout, not only in the analytical proofs but also in the few which are purely geometrical in form. It need not be said that, if we are building on the foundation of the ordinary real geometry, say, as contained in Euclid, much is required in the way of definition and deduction before proofs of the latter kind can be considered complete, and unfortunately it is still the practice in our elementary text-books to leave this to be supplied by the reader. Partly to fill some of the blanks, but chiefly to put in relief the point of view from which the subject is considered, one or two paragraphs dealing with the most elementary matters have been inserted at the commencement of the paper.

The ambiguity of certain elements associated with a circle or a system of circles, as, *e.g.*, the radius of a circle, the common tangent of two circles, the axis of similitude of three circles, is the source of inconvenience in the statement of many general theorems, and an attempt has been made to remove this ambiguity by laying down suitable definitions. As one application the inversion invariant of two circles is investigated and a current error pointed out in the statement of Casey's important extension of Ptolemy's Theorem.

Several methods are given for determining the centres and radii of the circles touching three given circles. Two generalisations of the problem are also discussed, and the circles are found which have given common tangents with three given circles, or which intersect

them at given angles. The solution of the latter problem is based upon a relation, which appears not to have been noticed before, between the angles of intersection of any five circles. This relation leads directly to

(i) the equation of the two circles satisfying the conditions,

(ii) the equation for the radii of these circles,

(iii) the equation of either circle in a form involving its radius.

The single equation of the two circles is a homogeneous quadratic in what may be called *tricircular coordinates*, any such coordinate being the square of the tangent from the variable point to one of the given circles, divided by the radius of that circle, and the equation involves, besides these coordinates, only the angles of intersection of the given circles with each other and with the required circles.

Several particular cases are worked out, as, for instance, the equations of the four pairs of circles which go through the points of intersection of the given circles, and the equations of the four pairs of Hart circles, each pair of which touches every Apollonian pair.

It is proved that the four pairs of circumscribing circles are also touched by other four pairs of circles. The corresponding proposition in spherical geometry is easy to prove, and perhaps known, but what suggested the theorem was a certain result due, I think, to Cayley and given by Salmon,* in the theory of conics having double contact with a given conic. In fact, the equation in tricircular coordinates of a pair of circles inverse to each other with respect to a given circle is identical in form with the equation in trilinears of a conic touching a given conic at two points. The relation between the theory of such conics and that of circles on a sphere has been noticed and used to advantage by Casey.* Some of the aspects of the connection between the three theories I hope to consider in a supplement to the present paper.

I have to thank my friend, Dr Muirhead, for his kindness in looking up some references, and in placing at my disposal a collection of abstracts of papers on the subject which he drew up for his own use some years ago.

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* Salmon's *Conic Sections*, Sixth Edition, Arts. 386, 387.

## Section I.

### *Definitions and Theorems.*

2. (*a*) *A right line* is the assemblage of pairs of values $(x, y)$, or, as we say, of *points* $(x, y)$, satisfying an equation of the first degree.

(*b*) Two lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$ are *parallel* if $AB' - A'B = 0$; they are *at right angles*, or *perpendicular*, to each other if $AA' + BB' = 0$.

(*c*) The *square of the distance* between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

The distance itself is two-valued, but we shall distinguish beween PQ and QP so as to have $PQ + QP = 0$, and, if P, Q, R are in a line, $PQ + QR + RP = 0$, as follows.

In the first place, if PQ is parallel to one of the axes, say, to $Ox$, we take $PQ = x_2 - x_1$.

The line through P, Q being $Ax + By + C = 0$, then, if $A^2 + B^2 = 0$, the formula gives $PQ^2 = 0$ and therefore $PQ = 0$.

In any other case, since $A(x_2 - x_1) + B(y_2 - y_1) = 0$, we have $(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_2 - x_1)^2(1 + A^2/B^2)$.

Fix upon one of the square roots of $1 + A^2/B^2$ and take
$$PQ = (x_2 - x_1)\sqrt{(1 + A^2/B^2)}.$$

If $R(x_3, y_3)$ and $S(x_4, y_4)$ lie on the line PQ or a parallel to it, take also
$$RS = (x_4 - x_3)\sqrt{(1 + A^2/B^2)},$$
and the product PQ . RS and ratio PQ : RS are definite, whichever square root of $1 + A^2/B^2$ we may have fixed upon.

(*d*) From Def. (*b*) the line through $P(x_1, y_1)$ perpendicular to $Ax + By + C = 0$, is
$$B(x - x_1) - A(y - y_1) = 0.$$

The *foot of the* $\perp^r$, N, satisfies both equations.

Writing $Ax + By + C = 0$ in the form
$$A(x - x_1) + B(y - y_1) = -(Ax_1 + By_1 + C)$$

we get by squaring and adding

$$(A^2+B^2)\{(x-x_1)^2+(y-y_1)^2\}=(Ax_1+By_1+C)^2$$

that is $$PN=\frac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}.$$

As in (*c*) one of the square roots is to be selected, and adhered to.

(*e*) If P is $(x_1, y_1)$, $Q(x_2, y_2)$ and $N(x_4, y_4)$ the equation of PN is

$$(x-x_1)(y_1-y_4)-(y-y_1)(x_1-x_4)=0$$

and of QN $$(x-x_2)(y_2-y_4)-(y-y_2)(x_2-x_4)=0.$$

PN, QN will therefore be at right angles, by (*b*), if

$$(x_1-x_4)(x_2-x_4)+(y_1-y_4)(y_2-y_4)=0,$$

which is equivalent to

$$(x_1-x_2)^2+(y_1-y_2)^2=(x_1-x_4)^2+(y_1-y_4)^2+(x_2-x_4)^2+(y_2-y_4)^2$$

or $PQ^2=PN^2+QN^2$. (Euc. I. 47).

Now let N be the foot of the $\perp^r$ from P to QR, where R is $(x_3, y_3)$, then, attending to (*c*)

$$\begin{aligned}2QN \,.\, QR &= QR^2+QN^2-(QR-QN)^2\\ &=QR^2+QN^2-NR^2\\ &=QR^2+(QN^2+NP^2)-(NR^2+NP^2)\\ &=QR^2+QP^2-PR^2. \quad \text{(Euc. II. 12, 13).}\end{aligned}$$

(*f*) A *circle* is the locus of points satisfying an equation of the form $$x^2+y^2+2gx+2fy+c=0.$$

When the equation is brought to the form

$$(x-\alpha)^2+(y-\beta)^2-r^2=0$$

$(\alpha, \beta)$ is the centre, and $r^2$ is the square of the radius.

The radius itself, which is $\sqrt{g^2+f^2-c}$ is two-valued. We fix on one of these values arbitrarily, and call that the radius $r$.

(*g*) The *centre of similitude* of two circles, radii $a$, $b$ and centres A, B is that point S in the line AB for which

$$SA:SB=a:b.$$

Hence if A is $(x_1, y_1)$ and $B(x_2, y_2)$, S is

$$\left(\frac{ax_2-bx_1}{a-b},\ \frac{ay_2-by_1}{a-b}\right).$$

For two real circles with positive radii, S is thus what is usually called the external centre of similitude. In the same case the internal centre of similitude is *the* centre of similitude of the circles $a$, $-b$, or of $-a$, $b$. Note that the centre of similitude of a circle, radius $a$ and the same circle, radius $-a$, is the centre of the circle. The centre of similitude of $a$, $a$ is indeterminate, that is, any point is a centre of similitude.

The *axis of similitude* of three circles $a$, $b$, $c$ is that line the $\perp^{rs}$ on which from the centres are proportional to the radii. It therefore passes through the three centres of similitude of the circles taken in pairs.

(*h*) Two circles of radii $a$, $b$, or, as we shall usually say for brevity, two circles $a$, $b$ will be said to *touch* if the square of the distance between the centres, $D^2$, is equal to $(a-b)^2$.

The circles have two coincident points in common not only when $D^2=(a-b)^2$ but also when $D^2=(a+b)^2$, but in the latter case we say that it is the circles $a$, $-b$ or $-a$, $b$ that *touch*. It may sometimes be convenient to state, in the ordinary sense, that two circles touch, but in such a case we shall take care that the radii are not specifically mentioned.

It follows from the definition that when two circles, with radii assigned, touch, the point of contact is the centre of similitude. Also, if two real circles touch internally (concavely), their radii have the same sign; if they touch externally (convexly), their radii have opposite signs.

(*i*) There are four lines which cut each circle in two coincident points.

The *common tangents* of $a$, $b$ are the two which pass through the centre of similitude.

The square of the length of a common tangent is defined to be $D^2-(a-b)^2$.

It vanishes when, and only when, the circles touch.

For the length itself, either root of its square may be taken.

($j$) The *cosine of the angle between two circles* $a$, $b$ is

$$\frac{a^2+b^2-D^2}{2ab}.$$

Here we follow Salmon, but for some purposes it would be a good deal more convenient to take the cosine with the opposite sign. For instance, the analogy between certain formulæ for circles, and the corresponding formulæ for right lines would thus be more apparent.

For two real intersecting circles, the cosine as defined belongs to the angle subtended at a common point by the line joining the centres.

For the angle itself, we fix on any one of the infinite number of angles whose cosines have the value specified.

The angle between two circles which touch may thus be taken as zero, the angle between a circle and itself as zero, and the angle between $a$ and $-a$ as $\pi$.

($k$) *The angle $\theta$ between a line* L *and a circle* $a$ is given by

$$p = a\cos\theta$$

where $p$ is the $\perp^r$ (of definite sign) from the centre to the line.

A line L *touches* a circle $a$ if it meets it at angle zero, that is, if $p = a$.

If the signs of all the $\perp^{rs}$ to the line L be changed, we may speak of the line in the altered circumstances as the line $-$L. If L touches $a$, then $-$L touches $-a$, but not $a$.

($l$) Two points P, Q are inverse to each other with respect to a circle of centre O and radius $k$, if O, P, Q are in a line and OP . OQ $= k^2$.

If O is the origin and P is $(x, y)$, Q is $\left(\dfrac{k^2x}{x^2+y^2}, \dfrac{k^2y}{x^2+y^2}\right)$.

The inverse of the circle

$$x^2+y^2+2gx+2fy+c=0 \qquad (1)$$

is thus the circle

$$x^2+y^2+2gxk^2/c+2fyk^2/c+k^4/c=0. \qquad (2)$$

Consider another pair of inverse circles

$$x^2+y^2+2g'x+2f'y+c'=0 \qquad (3)$$

and $$x^2+y^2+2g'xk^2/c'+2f'yk^2/c'+k^4/c'=0. \qquad (4)$$

Let the radii of the four circles be $r_1$, $r_2$, $r_3$, $r_4$; the angle between (1) and (3) $\theta_1$ and that between (2) and (4) $\theta_2$.

Then by ($j$) $2r_1r_3\cos\theta_1 = 2gg' + 2ff' - c - c'$,

$$2r_2r_4\cos\theta_2 = (2gg' + 2ff' - c - c')k^4/cc'.$$

Now $r_2 = \pm k^2r_1/c$ and $r_4 = \pm k^2r_3/c'$.

If we take $r_2 = +k^2r_1/c$ and $r_4 = +k^2r_3/c'$ we shall have $\cos\theta_1 = \cos\theta_2$; also according to ($g$) O will be the centre of similitude of each inverse pair.

Two circles will not be called inverse to each other, when their radii are definite, unless the centre of inversion is their centre of similitude.

Then we have the theorem that two circles cut at the *same* angle as their inverses.

By a remark made under ($g$) the circle of inversion $k$ is not inverse to itself but to the circle $-k$; also a circle will be inverse to itself if the square of the tangent from O is equal to $k^2$. Hence the circle of inversion $k$ cuts two inverse circles at *supplementary* angles; a self-inverse circle cuts them at equal angles.

($m$) If, in (1) of ($l$), $c = 0$, instead of (2) we have the line

$$2gx + 2fy + k^2 = 0.$$

We prove in the same way that $\cos\theta_1 = \cos\theta_2$ provided $k^2 = -2r_1p$, where $p$ is the $\perp^r$ from O on the line. The line will be said to be inverse to the circle (1) when the signs of its $\perp^{rs}$ are so determined that that relation is fulfilled.

($n$) Let $D_1$ be the distance between the centres of (1), (3) of ($l$); $D_2$ that between the centres of (2), (4).

Since $\cos\theta_1 = \cos\theta_2$ we have

$$\frac{D_1^2 - r_1^2 - r_3^2}{2r_1r_3} = \frac{D_2^2 - r_2^2 - r_4^2}{2r_2r_4}$$

and therefore $\dfrac{D_1^2 - (r_1 - r_3)^2}{2r_1r_3} = \dfrac{D_2^2 - (r_2 - r_4)^2}{2r_2r_4}$

or the quotient of the square of the common tangent of two circles by the product of their radii is not altered by inversion.

The statement of the theorem in the usual unconventional language is much less simple.

Thus let $a$, $b$, $c$, $d$ be the real positive radii of the real circles (1), (2), (3), (4).

To fix the ideas, we may suppose $k^2$ positive, but the final statement will not be affected if $k^2$ be negative. Then if O is outside (1) it will be the *external* centre of similitude of (1), (2), but if O is inside (1) it will be their *internal* centre of similitude. Similarly with (3), (4).

If O is outside both of (1), (3) then for the purposes of the theorem the radii of (2), (4) are $+b$, $+d$; if O is inside both of (1), (3) the radii of (2), (4) are $-b$, $-d$; it being supposed that the radii $a$, $c$ are kept fixed. In these cases, then, we have

$$\frac{D_1^2-(a-c)^2}{ac}=\frac{D_2^2-(b-d)^2}{bd}.$$

If we had taken the radius of (3) as $-c$, that of (4) would have been $-d$, and therefore

$$\frac{D_1^2-(a+c)^2}{ac}=\frac{D_2^2-(b+d)^2}{bd}.$$

These two relations, which obviously are not independent, show that if the centre of inversion is *inside all the circles, or outside them all*, then

(sq. of either com. tang.)/prod. of radii inverts into
(sq. of similar com. tang.)/prod. of radii.

But if O is outside one of the two original circles, and inside the other, the radii of the inverse circles will have opposite signs, and therefore

$$\frac{D_1^2-(a-c)^2}{ac}=-\frac{D_2^2-(b+d)^2}{bd}$$

$$\text{and}\quad \frac{D_1^2-(a+c)^2}{ac}=-\frac{D_2^2-(b-d)^2}{bd},$$

that is to say, if the centre of inversion is inside one inverse pair, but outside the other inverse pair, then

(sq. of either com. tang.)/prod. of radii inverts into
– (sq. of the dissimilar com. tang.)/prod. of radii.

In the case of two intersecting circles, it is easy to verify the result from a figure, by comparing the angles between the original and inverted circles.

In the first case it will be readily seen that $\cos\theta_1 = \cos\theta_2$, but in the second that $\cos\theta_1 = -\cos\theta_2$.

Casey, to whom the theorem is due, states simply that "if two circles be inverted into two others, the square of the common tangent of the first pair, divided by the rectangle contained by their diameters, is equal to the square of the common tangent of the second pair, divided by the rectangle contained by their diameters." This is vague, but from the way in which he applies the theorem it seems clear that he understands the common tangent to be direct in both cases, or else transverse in both cases.

He is thus led into an inaccurate statement of his theorem

$$12 \cdot 34 \pm 13 \cdot 24 \pm 14 \cdot 23 = 0$$

respecting the common tangents of four circles touched by a fifth. He states that the direct tangent is to be taken between two circles which are *on the same side* of the fifth circle, the transverse between two which are on opposite sides of it. This is wrong and ought to be that the direct tangent is taken when two circles are both touched concavely, or both convexly, by the fifth circle; the transverse when one is touched concavely and the other convexly.* Two circles touched convexly are necessarily on the same side of the fifth circle, namely, the outside; but a circle touched concavely may be either outside or inside of it. Salmon, in giving an account of the matter, uses different methods from Casey, but repeats the defective enunciation of the theorem. (Salmon's *Conic Sections*, last 3 Arts. of Chap. VIII., Sixth Edition.)

## SECTION II.

### *First solution of the Apollonian and allied problems.*

3. *To find the centres and radii of the circles touching three given circles a, b, c.*

We do not assume that $a$, $b$, $c$ are positive, even if they are real.

*First Method.*

Let $D(x_1, y_1)$, $E(x_2, y_2)$, $F(x_3, y_3)$ be the centres of $a$, $b$, $c$.

We take the origin at the radical centre O so that

$$x_1^2 + y_1^2 - a^2 = x_2^2 + y_2^2 - b^2 = x_3^2 + y_3^2 - c^2 = k^2.$$

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* See Art. 16 below.

The circle, centre O and radius $k$, will be referred to throughout as the *orthogonal circle* simply.

Let $P(\xi, \eta)$ be the centre of a circle $\rho$ touching $a$, $b$, $c$.

Then from Art. 2 ($h$) we have three equations, of which the first is

$$(\xi - x_1)^2 + (\eta - y_1)^2 = (\rho - a)^2$$

$$\text{or}\quad 2\xi x_1 + 2\eta y_1 + (\rho^2 - \xi^2 - \eta^2 - k^2) = 2a\rho. \qquad (1)$$

This and the two similar equations may be regarded as asserting that the $\perp^{rs}$ from $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ on the line

$$2\xi x + 2\eta y + (\rho^2 - \xi^2 - \eta^2 - k^2) = 0 \qquad (2)$$

are as $a : b : c$.

The line (2) is therefore the axis of similitude.

Let the actual $\perp^{rs}$ on it from D, E, F, O be $a/\lambda$, $b/\lambda$, $c/\lambda$, $p$.

Then by 2 ($d$) $\quad 2a\rho : a/\lambda = \rho^2 - \xi^2 - \eta^2 - k^2 : p$

that is
$$2\lambda p\rho = \rho^2 - \xi^2 - \eta^2 - k^2 \qquad (3)$$
and
$$\lambda^2 p^2 = \xi^2 + \eta^2. \qquad (4)$$

Eliminating $\xi^2 + \eta^2$ from (3), (4) we get the equation for $\rho$, viz.,

$$(\lambda^2 - 1)\rho^2 + 2\lambda p\rho + k^2 = 0. \qquad (5)$$

The line OP which is $\xi y - \eta x = 0$, is $\perp^r$ to (2), the axis of similitude.

The $\perp^r$ $p_1$ from P on (2) is

$$(\rho^2 + \xi^2 + \eta^2 - k^2)/2\lambda\rho$$

and if $OP = \sigma$, taken with definite sign as a segment on the same line as $p$ and $p_1$,

$$\begin{aligned} \sigma &= p - p_1 \\ &= -(\xi^2 + \eta^2)/\lambda p \\ &= -\lambda p. \end{aligned} \qquad (6)$$

We have thus defined precisely the centres and radii of two circles $\rho_1$, $\rho_2$ touching $a$, $b$, $c$.

Since $\sigma/\rho$ is the same for both, O is their centre of similitude.

Also from (3) which is $\quad 2\sigma p = \sigma^2 + k^2 - \rho^2$

it follows from Euclid II., 12, 13 proved in 2 ($e$) that the points of intersection of $\rho$, $k$ are on the axis of similitude.

The circles $\rho_1$, $\rho_2$ are therefore inverse to each other with respect to $k$, and the axis of similitude is their radical axis. The first part of this statement is proved more simply by observing that the

inverse of $\rho_1$ with respect to $k$ will, by 2 ($l$), cut $a$, $b$, $c$ which are self-inverses at the same angle as $\rho_1$, that is, will also touch them.

Again if X is the point of contact of $\rho$, $a$; and if the $\perp^r$ from D to the axis of similitude meets it at L, and OX at G; we have from the similar triangles OPX, GDX, attending to signs

$$\mathrm{OP}:\mathrm{GD}=\mathrm{XP}:\mathrm{XD}$$
$$=\rho \quad : a \quad , \text{ by } 2\,(g),\,(h).$$
$$\therefore \quad \mathrm{GD}=a\sigma/\rho=-\lambda a, \text{ by } (6).$$
$$\text{Also} \quad \mathrm{DL}=a/\lambda\,;$$

$\therefore$ $\mathrm{DL}\,.\,\mathrm{DG}=a^2$, and G is the pole of the axis of similitude with respect to the circle $a$. Hence Gergonne's construction, viz., we find $X_1$, $X_2$ as the intersections of $a$ with OG; then $P_1$, $P_2$ are the intersections of the $\perp^r$ to the axis of similitude through O with $X_1D$, $X_2D$.

4. *Second Method.*

If X, Y be the centres of two circles $x$, $y$;
$T_1^2$, $T_2^2$ the squares of the tangents from a point U, that is

$$T_1^2=\mathrm{UX}^2-x^2, \quad T_2^2=\mathrm{UY}^2-y^2,$$

and if UN be the $\perp^r$ from U to the radical axis, we have the fundamental theorem

$$T_1^2-T_2^2=2\mathrm{UN}\,.\,\mathrm{YX}.$$

Apply the theorem to the circles $k$, $\rho$ taking U, first at D, then at O, and putting $p_a$, $p$ for the $\perp^{rs}$ from D, O on the radical axis of $k$, $\rho$.

Thus
$$\left.\begin{aligned} a^2-\{(a-\rho)^2-\rho^2\} &= 2p_a(-\sigma) \\ -k^2-(\sigma^2-\rho^2) &= 2p\,(-\sigma) \end{aligned}\right\}$$
and

that is
$$\left.\begin{aligned} a\rho &= -\sigma p_a \\ \sigma^2+k^2-\rho^2 &= 2\sigma p \end{aligned}\right\}.$$

The first of these, and the two similar equations

$$b\rho=-\sigma p_b, \quad c\rho=-\sigma p_c$$

show that the radical axis of $k$, $\rho$ is the axis of similitude of $a$, $b$, $c$; and if we put $p_a=a/\lambda$ we have the two equations

$$\left.\begin{aligned} \sigma &= -\lambda\rho \\ \sigma^2+k^2-\rho^2 &= 2\sigma p \end{aligned}\right\} \text{ and the whole theory as before.}$$

5. *Third Method.*

Take P an undefined point on the $\perp^r$ from O to the axis of similitude; $OP=\sigma$; $\perp^{rs}$ from O, D to axis of similitude $=p,\ a/\lambda$.

By Euclid II. 12, 13

$$PD^2=OD^2+OP^2-2OP(p-a/\lambda)$$
$$=a^2+k^2+\sigma^2-2\sigma p+2\sigma pa/\lambda.$$

We shall have

$$PD^2=(a-\rho)^2$$
$$\text{if}\quad 0=(k^2+\sigma^2-2\sigma p-\rho^2)+2a(\rho+\sigma/\lambda).$$

We shall therefore also have

$$PE^2=(b-\rho)^2 \quad\text{and}\quad PF^2=(c-\rho)^2$$

if we take

$$\left.\begin{array}{ll} k^2+\sigma^2-2\sigma p-\rho^2 & =0 \\ \rho+\sigma/\lambda & =0 \end{array}\right\}.$$

6. The last method may be used to solve a more general problem, viz., if the radii of the three circles with centres D, E, F are changed to $a-x$, $b-x$, $c-x$, to find the new axis of similitude and the new orthogonal circle.

We assert in the first place that as $x$ varies the axis of similitude moves parallel to itself, and the radical centre in a line $\perp^r$ to it. This is obvious if we assume the results just proved, for the centres of the circles touching $a-x$, $b-x$, $c-x$ will remain fixed, their radii being $\rho_1-x$, $\rho_2-x$. For an independent proof, note first that the parallel to the axis of similitude of $a$, $b$, $c$ at distance $-x/\lambda$ from it will be at distances $(a-x)/\lambda$, $(b-x)/\lambda$, $(c-x)/\lambda$ from D, E, F, and will therefore be the axis of similitude of $a-x$, $b-x$, $c-x$.

Next take O′ a point on the $\perp^r$ to the axis of similitude from O; $OO'=y$.

We have $\quad O'D^2=OD^2+OO'^2-2OO'(p-a/\lambda)$

$$=a^2+k^2+y^2-2yp+2ya/\lambda$$

and $\quad O'D^2-(a-x)^2=(k^2+y^2-2yp-x^2)+2a(x+y/\lambda).$

Hence if we define $y$ by the equation $x+y/\lambda=0$ we shall have

$$O'D^2-(a-x)^2=O'E^2-(b-x)^2=O'F^2-(c-x)^2$$
$$=(\lambda^2-1)x^2+2\lambda px+k^2;$$

$\therefore$ O$'$ is the radical centre of $a-x$, $b-x$, $c-x$ and the square of the radius of their orthogonal circle is given by

$$k'^2 = (\lambda^2-1)x^2 + 2\lambda px + k^2.$$

$k'$ will be zero when the circles have a common point which must be either $P_1$ or $P_2$.

The radius of the touching circle is then zero, *i.e.*, $x$ is equal to $\rho_1$ or $\rho_2$; hence the equations for $\sigma$, $\rho$ as before.

The new radical centre O$'$ will be on the new axis of similitude when

$$OO' = p - x/\lambda$$

$$\text{or} \quad (\lambda^2-1)x = -\lambda p.$$

Since with this value of $x$, the centre of similitude of the touching circles is on their radical axis, it is geometrically obvious that their radii are equal and of opposite signs, that is,

$$(\rho_1 - x) + (\rho_2 - x) = 0.$$

Of course this also follows immediately from the quadratic for $\rho$. The result will be used in Art. 13.

7. The equation $\quad (\lambda^2-1)\rho^2 + 2\lambda p\rho + k^2 = 0$
may be partially verified as follows.

(i) When $k^2 = 0$; the three circles have a common point, which is a touching circle of radius zero; the equation gives one value of $\rho = 0$.

(ii) When $\lambda = 1$; the $\perp^{rs}$ from the centres to the axis of similitude are $a$, $b$, $c$ and the axis of similitude is a common tangent.

When $\lambda = -1$; the $\perp^{rs}$ are $-a$, $-b$, $-c$ and the axis of similitude with the signs of all its $\perp^{rs}$ changed is a common tangent.

In either case one of the touching circles is a line; the equation gives one value of $\rho$ infinite.

(iii) When $p = 0$; the radical centre lies on the axis of similitude, and therefore as at the end of Art. 6, the radii of the touching circles are equal but of opposite signs; the equation gives $\rho_1 + \rho_2 = 0$.

8. Suppose that the three circles are real, and that the absolute values of their radii are $r_1$, $r_2$, $r_3$.

If in the above analysis we take $a=r_1$, $b=r_2$, $c=r_3$ the axis of similitude is that which has D, E, F all on one side of it. To a positive real root of the equation for $\rho$ corresponds a circle touching all the circles concavely; to a negative real root a circle touching them all convexly. (2 (*h*), last remark.)

If we take $a=-r_1$, $b=r_2$, $c=r_3$ the axis of similitude is that which has D on one side of it, and E, F on the other. A positive real value of $\rho$ gives a circle touching the first circle convexly, the other two concavely; a negative real value of $\rho$ gives a circle touching the first circle concavely, the other two convexly.

Similarly for the cases

$$a=r_1,\ b=-r_2,\ c=r_3 \quad \text{and} \quad a=r_1,\ b=r_2,\ c=-r_3.$$

Since $a$, $b$, $c$ have the same axis of similitude and orthogonal circle as $-a$, $-b$, $-c$, the other four permutations of signs of the radii do not yield further solutions.

A very persistent form of erroneous statement of these results should be noticed. For the first case, as an instance, it is frequently said that when the values of $\rho$ are real, a pair of tangent circles exists each of which has the three given circles *all on the same side of it.* The error is analogous to that which was noticed at the end of Art. 2.

9. *To find the centres and radii of the circles cutting $a$, $b$, $c$ at given angles $\alpha$, $\beta$, $\gamma$.*

We can proceed precisely as in Art. 3 till we come to equation (1) which will now be

$$2\xi x_1 + 2\eta y_1 + (\rho^2 - \xi^2 - \eta^2 - k^2) = 2a\rho\cos\alpha.$$

The $\perp^{\text{rs}}$ on the line (2) will therefore be as $a\cos\alpha : b\cos\beta : c\cos\gamma$.

This property defines that line; it cuts the circle at angles with cosines proportional to $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and we may call it the $\alpha$, $\beta$, $\gamma$ axis.

If the actual $\perp^{\text{rs}}$ on it from D, E, F, O are

$$a\cos\alpha/\lambda',\ b\cos\beta/\lambda',\ c\cos\gamma/\lambda',\ p',$$

we find equations of the same form as before

$$\left.\begin{array}{c} \sigma = -\lambda'\rho \\ (\lambda'^2 - 1)\rho^2 + 2\lambda' p'\rho + k^2 = 0 \end{array}\right\}.$$

The circles $\rho_1$, $\rho_2$ are inverse with respect to $k$, and the $\alpha$, $\beta$, $\gamma$ axis is their radical axis.

10. *To find the centres and radii of the circles the squares of whose common tangents with a, b, c have given values $u^2$, $v^2$, $w^2$.*

Following again the lines of Art. 3, we choose the origin O′ so that

$$x_1^2+y_1^2-a^2-u^2=x_2^2+y_2^2-b^2-v^2=x_3^2+y_3^2-c^2-w^2 \quad (=k'^2 \text{ say}).$$

O′ is therefore the radical centre of the circles with centres D, E, F and squares of radii $a^2+u^2$, $b^2+v^2$, $c^2+w^2$, and $k'^2$ is the square of the radius of their orthogonal circle.

The line (2), with $k'^2$ for $k^2$, is still the axis of similitude of $a$, $b$, $c$.

If the $\perp^r$ from O′ to it is $p''$, we have

$$\left.\begin{aligned}\sigma&=-\lambda\rho\\(\lambda^2-1)\rho^2+2\lambda p''\rho+k'^2&=0\end{aligned}\right\}.$$

The circles $\rho_1$, $\rho_2$ are inverses with respect to $k'$ and the axis of similitude of $a$, $b$, $c$ is their radical axis.

Either of the two problems just discussed may be reduced at once to the other, for

$$\rho^2-2a\rho\cos a+a^2=(\rho-a\cos a)^2+a^2\sin^2 a$$

and therefore a circle which cuts $a$, $b$, $c$ at angles $a$, $\beta$, $\gamma$ will have $a^2\sin^2 a$, $b^2\sin^2\beta$, $c^2\sin^2\gamma$ for the squares of its common tangents with the circles whose centres are D, E, F and radii $a\cos a$, $b\cos\beta$, $c\cos\gamma$.

The other methods given for the case of contact may easily be adapted to the more general problems.

11. The methods of this section fail in certain cases; notably when the centres of the circles are in a line, a case requiring exceptional treatment in most general methods for the contact problem, including Gergonne's. The other cases of failure arise when the axis of similitude or the $a$, $\beta$, $\gamma$ axis is at an infinite distance, but these cases are easily met. Thus in 3, when $a=b=c$, equations (1) give at once

$$\xi=0,\quad \eta=0\ ;\quad \rho^2-k^2=2a\rho.$$

In 9 when $a\cos a=b\cos\beta=c\cos\gamma$, we have similarly

$$\xi=0,\quad \eta=0\ ;\quad \rho^2-k^2=2a\rho\cos a.$$

The equations obtained in this section involve coefficients whose geometrical meaning, as we have seen, is very simple; but results may be required in terms of more fundamental constants of the system of given circles, as for instance, the radii and distances between the centres. It is not difficult, but certainly tedious, to deduce such results directly from those found here. One way is to

use trilinear coordinates with the triangle of centres as triangle of reference. In the investigation I found the following formula useful; it gives the trilinear coordinates $\alpha$, $\beta$, $\gamma$ of a point P in terms of the squares of its distances from the vertices A, B, C of the triangle of reference, whose sides are $a$, $b$, $c$.

$$4\Delta\alpha = -a\,.\,\mathrm{AP}^2 + b\cos\mathrm{C}\,.\,\mathrm{BP}^2 + c\cos\mathrm{B}\,.\,\mathrm{CP}^2 + abc\cos\mathrm{A}$$

with similar expressions for $\beta$ and $\gamma$.

But much more powerful methods are available.

## Section III.

*Miscellaneous methods for the Apollonian problem.*

12. *To find the radii of the circles touching* $a$, $b$, $c$.

It has been remarked by numerous writers that the relation between the six mutual distances of four points in a plane furnishes a natural and immediate solution of this problem.

If $\mathrm{PD}^2 = x^2$, $\mathrm{PE}^2 = y^2$, $\mathrm{PF}^2 = z^2$ and $\mathrm{EF}^2 = d^2$, $\mathrm{FD}^2 = e^2$, $\mathrm{DE}^2 = f^2$ then

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & f^2 & e^2 & x^2 \\ 1 & f^2 & 0 & d^2 & y^2 \\ 1 & e^2 & d^2 & 0 & z^2 \\ 1 & x^2 & y^2 & z^2 & 0 \end{vmatrix} = 0. \quad (1).$$

To find the radii we have only to put $x^2 = (a-\rho)^2$, $y^2 = (b-\rho)^2$, $z^2 = (c-\rho)^2$. But the equation will be much simpler if we first transform this determinant D as follows.

From the 2nd, 3rd, and 4th columns subtract the last; then deal in the same way with the rows. Thus

$$\mathrm{D} = \begin{vmatrix} 2x^2, & x^2+y^2-f^2, & x^2+z^2-e^2 \\ x^2+y^2-f^2, & 2y^2, & y^2+z^2-d^2 \\ x^2+z^2-e^2, & y^2+z^2-d^2, & 2z^2 \end{vmatrix}.$$

Divide the rows by $2x$, $2y$, $2z$ and the columns by $x$, $y$, $z$ and we may write

$$\mathrm{D} = 8x^2y^2z^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & \dfrac{x^2+y^2-f^2}{2xy} & \dfrac{x^2+z^2-e^2}{2xz} \\ 1 & \dfrac{x^2+y^2-f^2}{2xy} & 1 & \dfrac{y^2+z^2-d^2}{2yz} \\ 1 & \dfrac{x^2+z^2-e^2}{2xz} & \dfrac{y^2+z^2-d^2}{2yz} & 1 \end{vmatrix}$$

and on subtracting the first row from each of the others

$$D = -8x^2y^2z^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & \frac{n^2}{2xy} & \frac{m^2}{2xz} \\ 1 & \frac{n^2}{2xy} & 0 & \frac{l^2}{2yz} \\ 1 & \frac{m^2}{2xz} & \frac{l^2}{2yz} & 0 \end{vmatrix} = -2 \begin{vmatrix} \frac{1}{2} & x & y & z \\ x & 0 & n^2 & m^2 \\ y & n^2 & 0 & l^2 \\ z & m^2 & l^2 & 0 \end{vmatrix}, \quad (2)$$

where $\quad l^2 = d^2 - (y-z)^2,\ m^2 = e^2 - (z-x)^2,\ n^2 = f^2 - (x-y)^2.$

If now we put $a-\rho$ for $x$, $b-\rho$ for $y$, $c-\rho$ for $z$, so that $l^2$, $m^2$, $n^2$ are now the squares of the common tangents of $a$, $b$, $c$, we find

$$\begin{vmatrix} \frac{1}{2} & a-\rho & b-\rho & c-\rho \\ a-\rho & 0 & n^2 & m^2 \\ b-\rho & n^2 & 0 & l^2 \\ c-\rho & m^2 & l^2 & 0 \end{vmatrix} = 0. \quad (3).$$

This determinant is of a familiar type and its expansion gives

$$l^2m^2n^2$$
$$+\,l^4(a-\rho)^2 + m^4(b-\rho)^2 + n^4(c-\rho)^2$$
$$-\,2m^2n^2(b-\rho)(c-\rho) - 2n^2l^2(c-\rho)(a-\rho) - 2l^2m^2(a-\rho)(b-\rho) = 0$$

or $\quad \rho^2(l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2)$

$$+\,2\rho\{(b+c)m^2n^2 + (c+a)n^2l^2 + (a+b)l^2m^2 - al^4 - bm^4 - cn^4\}$$
$$+\,(l^2m^2n^2 + a^2l^4 + b^2m^4 + c^2n^4 - 2bcm^2n^2 - 2can^2l^2 - 2abl^2m^2) = 0. \quad (4)$$

This is the equation found geometrically by Mr Alex. Holm in last year's *Proceedings*, except that, in accordance with our conventions, we have $-\rho$ instead of his $x$.

13. *Deduction of the equation found in Section II.*

If we write the left sides of equations (3), (4) just found as

$$L\rho^2 + M\rho + N,$$

L, M, N must from Art. 3 be proportional to $\lambda^2 - 1$, $2\lambda p$, $k^2$.

This we shall now prove independently by a development of the remarks in 7.

(i) The circles with centres D, E, F and squares of radii $a^2+k^2$, $b^2+k^2$, $c^2+k^2$ have a common point, viz., the radical centre of $a$, $b$, $c$.

In 12 (1) put $x^2=a^2+k^2$, $y^2=b^2+k^2$, $z^2=c^2+k^2$. Then from last column and last row subtract the first $\times k^2$. Thus

$$-2k^2\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & f^2 & e^2 \\ 1 & f^2 & 0 & d^2 \\ 1 & e^2 & d^2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & f^2 & e^2 & a^2 \\ 1 & f^2 & 0 & d^2 & b^2 \\ 1 & e^2 & d^2 & 0 & c^2 \\ 1 & a^2 & b^2 & c^2 & 0 \end{vmatrix} = 0.$$

The first determinant is $-16\Delta^2$ where $\Delta =$ area of $\Delta$ DEF; the second, by the transformation by which (2) was derived from (1) in 12, is $-2$N.

$$\therefore\quad \mathrm{N}=k^2 . 16\Delta^2.$$

(ii) The circles with centres D, E, F and radii $a/\lambda$, $b/\lambda$, $c/\lambda$ have a common tangent. Hence if we put $a/\lambda$ for $a$, and so on, in (3) or (4) above the coefficient of the highest power of $\rho$ must vanish, that is

$$l'^4+m'^4+n'^4-2m'^2n'^2-2n'^2l'^2-2l'^2m'^2=0$$

where $\qquad l'^2=d^2-(b-c)^2/\lambda^2$, etc.

Write the left side of this equation $\mathrm{A}+\mathrm{B}/\lambda^2+\mathrm{C}/\lambda^4$.

Then $\mathrm{C}=0$, for it is $-16\times$ square of area of $\Delta$ with sides $b-c$, $c-a$, $a-b$;

$\mathrm{A}=-16\Delta^2$, by putting $1/\lambda=0$.

To get B, note that when $\lambda=1$, $l'^2=l^2$ and therefore $\mathrm{A}+\mathrm{B}=\mathrm{L}$; but $\mathrm{A}+\mathrm{B}/\lambda^2=0$; hence $\lambda^2-1=-(\mathrm{A}+\mathrm{B})/\mathrm{A}$,

*i.e.*, $\quad \mathrm{L}=(\lambda^2-1) . 16\Delta^2$.

(iii) By Art. 6 if we take $a-x$, $b-x$, $c-x$ for radii the coefficient of $\rho$ in the resulting quadratic will vanish when

$$x(\lambda^2-1)=-\lambda p.$$

But when $a$, $b$, $c$ are replaced by $a-x$, $b-x$, $c-x$ in 12 (4) the coefficient of $\rho$ becomes $\mathrm{L}x+\frac{1}{2}\mathrm{M}$. $\therefore$ $\mathrm{L}.2\lambda p=\mathrm{M}(\lambda^2-1)$, and, using (ii),

$$\mathrm{M}=2\lambda p . 16\Delta^2.$$

14. *The equation for $\rho$ deduced from considerations of Solid Geometry.*

Considered analytically, the problem of finding a circle, centre $P(\xi, \eta)$, radius $\rho$, to touch the circles,

centres $D(x_1, y_1)$, $E(x_2, y_2)$, $F(x_3, y_3)$ and radii $a$, $b$, $c$ is the same as that of solving the three equations of the form

$$(\xi - x_1)^2 + (\eta - y_1)^2 = (\rho - a)^2.$$

These equations express that the point in space $(\xi, \eta, i\rho)$ is at distance zero from each of the points $(x_1, y_1, ia)$, etc. The problem is therefore equivalent to that of finding the centre of a sphere of given radius passing through three given points, and this can be solved geometrically.

In order to obtain a figure with as many real elements as possible, we shall suppose that the radii are all pure imaginaries, but the formula obtained for $\rho$ will still be true even if $a$, $b$, $c$ are all real.

Let L, M, N be the points $(x_1, y_1, ia)$, etc.

Then $\quad MN^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 - (b - c)^2 = l^2$;

$NL^2 = m^2$; $LM^2 = n^2$.

If H (X, Y, Z) is the centre and R the radius of the circumcircle of LMN, then a point Q $(\xi, \eta, i\rho)$ at zero distance from L, M, N is on the normal through H to the plane LMN at distance $\pm i$R from H.

A well-known formula gives the coordinates of H in terms of those of L, M, N.

In particular,

$$Z = \frac{l\cos L \,.\, ia + m\cos M \,.\, ib + n\cos N \,.\, ic}{l\cos L + m\cos M + n\cos N}$$

$$= \{l^2(m^2 + n^2 - l^2)a + m^2(n^2 + l^2 - m^2)b + n^2(l^2 + m^2 - n^2)c\}i/16\Delta_{lmn}^2.$$

Then, for the $z$ coordinate of Q,

$$i\rho = Z \pm i R\cos\phi$$

where $\phi$ is the angle between the planes LMN, DEF, and therefore

$$\cos\phi = \Delta_{def}/\Delta_{lmn}.$$

$$\text{Also} \quad R = lmn/4\Delta_{lmn}.$$

Hence $\rho = \{l^2(m^2 + n^2 - l^2)a + \ldots + \ldots \pm 4lmn\Delta_{def}\}/16\Delta_{lmn}^2.$

This is the formula obtained by Mr Holm in his paper of last year.

We can also derive the results of Art. 3.

(i) If $p_1$ is the $\perp^r$ DT from D to the line of intersection ST of the planes LMN, DEF
then $ia = p_1\tan\phi$ ; similarly $ib = p_2\tan\phi$ and $ic = p_3\tan\phi$.

Hence ST is the axis of similitude, and $\tan\phi = i\lambda$.

$$\therefore \quad \lambda^2 - 1 = -\sec^2\phi$$

$$\text{and} \quad (\lambda^2 - 1)\Delta^2_{def} = -\Delta^2_{lmn}.$$

(ii) Let the normal HQ to the plane LMN meet the plane DEF in K, and let KUS be $\perp^r$ to ST.
Then UH $\perp^r$ to KUS is Z.

Also $\quad KD^2 - a^2 = KD^2 + DL^2 = KH^2 + R^2.$

$$\therefore \quad KD^2 - a^2 = KE^2 - b^2 = KF^2 - C^2.$$

$\therefore$ K is the radical centre and $k^2 = KH^2 + R^2$

$$= Z^2\sec^2\phi + R^2.$$

(iii) KS is $p$, and $Z = p\sin\phi\cos\phi = p\tan\phi\cos^2\phi$
so that $\quad (\lambda^2 - 1)Z = -i\lambda p.$

Now $\quad \rho = -iZ \pm R\cos\phi$

and the quadratic for $\rho$ is

$$(\rho + iZ)^2 = R^2\cos^2\phi,$$

that is $\quad \rho^2 + 2iZ\rho - (Z^2 + R^2\cos^2\phi) = 0,$

or, multiplying by $\lambda^2 - 1$, which is equal to $-\sec^2\phi$,

$$(\lambda^2 - 1)\rho^2 + 2i(\lambda^2 - 1)Z\rho + Z^2\sec^2\phi + R^2 = 0$$

$$i.e., \quad (\lambda^2 - 1)\rho^2 + 2\lambda p\rho + k^2 = 0.$$

It is also obvious from the figure that the centre P of $\rho$ lies on KS and that $\sigma = -\lambda\rho$.

## Section IV.

*Application of general theorems relating to given circles.*

*Equations of the circles required by the general problems, in terms of tricircular coordinates.*

15. We write $S_1$ for $x^2 + y^2 + 2g_1x + 2f_1y + c_1$,

$S_2$ for $x^2 + y^2 + 2g_2x + 2f_2y + c_2$, and so on.

$S_1$ is the square of the tangent from $(x, y)$ to the circle $S_1 = 0$.

Also we shall habitually use $S_1$, $S_2$, $S_3$ with reference to the three circles $a$, $b$, $c$ round which the problems hinge.

When we speak of inverse points, or inverse circles, we shall understand the circle of inversion to be the orthogonal circle $k$.

The values of $S_1$, $S_2$, $S_3$ at the point $(x, y)$ are proportional to their values at the inverse point $(x', y')$; for, taking the origin at the radical centre, we find at once

$$S_1(x', y') = S_1(x, y) \,.\, k^2/(x^2 + y^2), \text{ etc., since } c_1 = c_2 = c_3 = k^2.$$

Hence a *homogeneous* equation in $S_1$, $S_2$, $S_3$ represents a locus which, if it contains any point, contains the inverse point also.

As an immediate consequence, a homogeneous equation of the first degree in $S_1$, $S_2$, $S_3$ represents a self-inverse circle. Such a circle is cut orthogonally by $k$; it is co-orthogonal with $a$, $b$, $c$, and for brevity we shall call it an *orthogonal* simply.

When the ratios $S_1 : S_2 : S_3$ are given, a pair of inverse points are determined, namely, the intersections of two orthogonals as

$$S_2 = C_1 S_1, \quad S_3 = C_2 S_1.$$

When a homogeneous equation of the second degree in $S_1$, $S_2$, $S_3$ is known to represent a circle as part of the locus, the complete locus must be this circle and its inverse.

It will often be convenient to write X, Y, Z instead of $S_1/a$, $S_2/b$, $S_3/c$.

The ratios of the *tricircular coordinates* X, Y, Z determine an *inverse pair* of points.

16. *The circles having given common tangents with $a$, $b$, $c$.*

By an application of a remarkably powerful method, due to Cayley, Salmon proves the relation between the common tangents 12, etc., of any five circles

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 12^2 & 13^2 & 14^2 & 15^2 \\ 1 & 12^2 & 0 & 23^2 & 24^2 & 25^2 \\ 1 & 13^2 & 23^2 & 0 & 34^2 & 35^2 \\ 1 & 14^2 & 24^2 & 34^2 & 0 & 45^2 \\ 1 & 15^2 & 25^2 & 35^2 & 45^2 & 0 \end{vmatrix} = 0. \qquad (1)$$

For 1, 2, 3 take the circles $a$, $b$, $c$; for 4 a circle $\rho$ the squares of whose common tangents with them are $u^2$, $v^2$, $w^2$.

The centre square of the determinant is then

$$\begin{matrix} 0 & n^2 & m^2 & u^2 \\ n^2 & 0 & l^2 & v^2 \\ m^2 & l^2 & 0 & w^2 \\ u^2 & v^2 & w^2 & 0 \end{matrix} \qquad - \quad - \quad - \quad - \quad (2)$$

This will remain unchanged in many of the manipulations now to be made, and we need then only write the border constituents.

Note in passing that if the circles 1, 2, 3, 4 are touched by a circle, say 5, then $15^2 = 25^2 = 35^2 = 45^2 = 0$ and (1) then reduces to, its centre square $= 0$, which is equivalent to

$$12 \cdot 34 \pm 13 \cdot 24 \pm 14 \cdot 23 = 0.^{*} \qquad - \quad - \quad - \quad (3)$$

Now two real circles with radii of the same sign must, by 2 (*h*), be touched similarly, *i.e.*, both concavely, or both convexly, by a real circle touching them both ; but dissimilarly when the radii are of opposite signs. Hence the proof of the statement at the end of Art. 2 that in (3) direct tangents are to be taken between circles touched similarly, transverse between circles touched dissimilarly, by the fifth circle.

Returning to (1), (2), for the fifth circle take the circle with centre $(x, y)$ and radius $r$.

Then $$15^2 = (x - x_1)^2 + (y - y_1)^2 - (r - a)^2$$
$$= S_1 + 2ar - r^2, \text{ etc.}$$

Substitute these values of $15^2, 25^2, \ldots,$ in (1) ; to the last row add $r^2$. the first, and similarly with the last column. We have then

$$\left| \begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & S_1 + 2ar \\ 1 & & & & & S_2 + 2br \\ 1 & & & & & S_3 + 2cr \\ 1 & & & & & S_4 + 2\rho r \\ 1, & S_1 + 2ar, & S_2 + 2br, & S_3 + 2cr, & S_4 + 2\rho r, & 2r^2 \end{array} \right| = 0. \quad (3)$$

* Conversely, if (3) is satisfied, the circles 1, 2, 3, 4 have a common tangent circle. To prove this, take the circle 5 so that $15^2 = 25^2 = 35^2 = 0$. The equation (1) is a quadratic for $45^2$, one of the roots of which is zero, in virtue of (3). Hence one of the two circles touching 1, 2, 3 touches 4 also.

This is an identical quadratic in $r$, and we obtain three useful equations by equating the absolute term, and the coefficients of $r$ and $r^2$ to zero.

(i) From the constant term

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & S_1 \\ 1 & & & & & S_2 \\ 1 & & & & & S_3 \\ 1 & & & & & S_4 \\ 1 & S_1 & S_2 & S_3 & S_4 & 0 \end{vmatrix} = 0 ; \quad (4)$$

but if $(x, y)$ is a point on the circle (4) then $S_4 = 0$; hence the equation of the two circles whose common tangents with $a$, $b$, $c$ are $u$, $v$, $w$, viz.,

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & & & & & S_1 \\ 1 & & & & & S_2 \\ 1 & & & & & S_3 \\ 1 & & & & & 0 \\ 1 & S_1 & S_2 & S_3 & 0 & 0 \end{vmatrix} = 0. \quad (5)$$

This equation is not homogeneous in the S's, but if from the last row and column we subtract the last but one, the equation is homogeneous in

$$S_1 - u^2, \ S_2 - v^2, \ S_3 - w^2 ;$$

showing that the two circles are inverse to each other with respect to the orthogonal circle of

$$S_1 - u^2 = 0, \quad S_2 - v^2 = 0, \quad S_3 - w^2 = 0.$$

If $u^2 = v^2 = w^2 = 0$, we have Casey's equation for the two touching circles

$$\begin{vmatrix} 0 & n^2 & m^2 & S_1 \\ n^2 & 0 & l^2 & S_2 \\ m^2 & l^2 & 0 & S_3 \\ S_1 & S_2 & S_3 & 0 \end{vmatrix} = 0. \quad - \quad - \quad - \quad (6)$$

(ii) From the coefficient of $r^2$ in (3)

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & a \\ 1 & & & & & b \\ 1 & & & & & c \\ 1 & & & & & \rho \\ 0 & a & b & c & \rho & \frac{1}{2} \end{vmatrix} = 0. \quad (7)$$

For the touching circles, put $u^2 = v^2 = w^2 = 0$; then from 2nd, 3rd and 4th rows subtract the 5th, and similarly with columns, and we obtain the equation already found in Art. 12.

(iii) From the coefficient of $r$ in (3)
$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & a \\ 1 & & & & & b \\ 1 & & & & & c \\ 1 & & & & & \rho \\ 1 & S_1 & S_2 & S_3 & S_4 & 0 \end{vmatrix} = 0, \quad (8)$$
and if $(x, y)$ is a point on the circle 4, the equation obtained by writing 0 instead of $S_4$ in (8) gives us individually the two circles required, $\rho$ being one of the roots of (7).

For the touching circle of radius $\rho$ put $u^2 = v^2 = w^2 = 0$, and from the 2nd, 3rd, 4th, and 6th rows subtract the 5th, and obtain
$$\begin{vmatrix} 0 & n^2 & m^2 & a-\rho \\ n^2 & 0 & l^2 & b-\rho \\ m^2 & l^2 & 0 & c-\rho \\ S_1 & S_2 & S_3 & -\rho \end{vmatrix} = 0, \quad (9)$$
or
$$\begin{aligned} & l^2S_1\{m^2(b-\rho) + n^2(c-\rho) - l^2(a-\rho)\} \\ & + m^2S_2\{n^2(c-\rho) + l^2(a-\rho) - m^2(b-\rho)\} \\ & + n^2S_3\{l^2(a-\rho) + m^2(b-\rho) - n^2(c-\rho)\} + 2l^2m^2n^2\rho = 0. \end{aligned} \quad (10)$$

The equation (6) of the two touching circles together is
$$l^4S_1^2 + \ldots + \ldots - 2m^2n^2S_2S_3 - \ldots - \ldots = 0.$$
Where this meets $S_1 = 0$, we have $m^2S_2 - n^2S_3 = 0$,
and therefore from (10) $S_2 = n^2\rho/(\rho - a)$ ; $S_3 = m^2\rho/(\rho - a)$.

Hence the points of contact of the two touching circles are separated, for obviously when the actual values of $S_1$, $S_2$, $S_3$ are given and not merely their ratios, a single point is determined.

As an interesting special case, suppose that two of the given circles, say $b$ and $c$, touch.

Then $l^2 = 0$ and the equation of the two touching circles is
$$(m^2S_2 - n^2S_3)^2 = 0.$$
The two touching circles coincide, and touch $b$, $c$ at their point of contact.

17. *The circles cutting a, b, c at given angles.*

The solution just given was deduced directly from the relation between the common tangents of five circles. The solution of the problem now proposed comes equally naturally from a similar relation connecting the cosines of their angles of intersection. To obtain this, let $(x_1, y_1)$ be the centre and $r_1$ the radius of the first circle, and so on.

Multiply together the two matrices

$$\begin{vmatrix} x_1^2+y_1^2-r_1^2, & -2x_1, & -2y_1, & 1 \\ x_2^2+y_2^2-r_2^2, & -2x_2, & -2y_2, & 1 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \quad \begin{vmatrix} 1, & x_1, & y_1, & x_1^2+y_1^2-r_1^2 \\ 1, & x_2, & y_2, & x_2^2+y_2^2-r_2^2 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

There being five rows and only four columns, the resulting determinant is zero, that is

$$\begin{vmatrix} -2r_1^2, & -2r_1r_2\cos 12, & -2r_1r_3\cos 13, & -2r_1r_4\cos 14, & -2r_1r_5\cos 15 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

or dividing the rows and columns in order by $r_1$, $r_2$, $r_3$, $r_4$, $r_5$,

$$\begin{vmatrix} 1 & \cos 12 & \cos 13 & \cos 14 & \cos 15 \\ \cos 12 & 1 & \cos 23 & \cos 24 & \cos 25 \\ \cos 13 & \cos 23 & 1 & \cos 34 & \cos 35 \\ \cos 14 & \cos 24 & \cos 34 & 1 & \cos 45 \\ \cos 15 & \cos 25 & \cos 35 & \cos 45 & 1 \end{vmatrix} = 0. \qquad (1)$$

This relation is of even greater interest than that between the common tangents. For one thing, it can be transferred at once by inversion to circles on a sphere, subject to a certain convention, or definition. Indeed in the whole of the following theory we might, with very slight changes, be dealing with circles on a sphere instead of on a plane.

For 1, 2, 3 take the circles $a$, $b$, $c$ cutting at angles A, B, C and for 4 take a circle $\rho$ cutting these at angles $\alpha$, $\beta$, $\gamma$. For 5 take the circle with centre $(x, y)$ and radius $r$.

Then
$$S_1 = (x-x_1)^2 + (y-y_1)^2 - a^2$$
$$= r^2 - 2ar\cos 15 \quad - \quad - \quad - \quad - \quad - \quad - \qquad (2)$$

or
$$\cos 15 = (r^2 - S_1)/2ar, \text{ and so on.}$$

Substitute in (1) and multiply the last row and column by $r$.

Thus
$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & (r^2-S_1)/2a \\ \cos C & 1 & \cos A & \cos\beta & (r^2-S_2)/2b \\ \cos B & \cos A & 1 & \cos\gamma & (r^2-S_3)/2c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & (r^2-S_4)/2\rho \\ \dfrac{r^2-S_1}{2a} & \dfrac{r^2-S_2}{2b} & \dfrac{r^2-S_3}{2c} & \dfrac{r^2-S_4}{2\rho} & r^2 \end{vmatrix} = 0. \qquad (3)$$

The coefficients of $r^4$ and $r^2$, and the constant term are separately zero.

(i) Putting $r=0$ and taking $(x, y)$ on the circle $S_4=0$, we have *the equation of the two circles cutting a, b, c at angles* $\alpha$, $\beta$, $\gamma$ in the form

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & X \\ \cos C & 1 & \cos A & \cos\beta & Y \\ \cos B & \cos A & 1 & \cos\gamma & Z \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 0 \\ X & Y & Z & 0 & 0 \end{vmatrix} = 0, \quad (4) \qquad \text{where } X = S_1/a,\ Y = S_2/b,\ Z = S_3/c.$$

From the 1st, 2nd, and 3rd rows subtract the 4th multiplied by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, and we get another form which is sometimes convenient

$$\begin{vmatrix} \sin^2\alpha & \cos C - \cos\alpha\cos\beta & \cos B - \cos\gamma\cos\alpha & X \\ \cos C - \cos\alpha\cos\beta & \sin^2\beta & \cos A - \cos\beta\cos\gamma & Y \\ \cos B - \cos\gamma\cos\alpha, & \cos A - \cos\beta\cos\gamma & \sin^2\gamma & Z \\ X & Y & Z & 0 \end{vmatrix} = 0. \quad (5)$$

By putting $\cos\alpha = \cos\beta = \cos\gamma = 1$ we get the equation of the touching circles in a form which is the same as 16 (6), seeing that $l^2 = 4bc\sin^2\frac{A}{2}$, etc.

(ii) From the coefficient of $r^4$ in (3)

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & 1/a \\ \cos C & 1 & \cos A & \cos\beta & 1/b \\ \cos B & \cos A & 1 & \cos\gamma & 1/c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 1/\rho \\ 1/a & 1/b & 1/c & 1/\rho & 0 \end{vmatrix} = 0. \quad (6)$$

This equation for the radii has been given by Salmon (*Conics*, Chap. IX., last example).

(iii) From the coefficient of $r^2$ in (3)

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & 1/a \\ \cos C & 1 & \cos A & \cos\beta & 1/b \\ \cos B & \cos A & 1 & \cos\gamma & 1/c \\ \cos\alpha & \cos\beta & \cos\gamma & 1 & 1/\rho \\ S_1/a & S_2/b & S_3/c & S_4/\rho & -2 \end{vmatrix} = 0. \quad (7)$$

This is the linear (non-homogeneous) relation connecting the equations of any four circles. If we take $(x, y)$ on $S_4=0$, we get the equations of the two circles cutting $a$, $b$, $c$ at angles $\alpha$, $\beta$, $\gamma$ in separate form. Also we might obtain (6) by putting the coefficient of $x^2+y^2=0$ in the identity (7).

18. *Some deductions from the equations of Art. 17.*

(i) From (1) if the circles 1, 2, 3, 4 are co-orthogonal, we may take 5 as their orthogonal circle and we get the determinant (D say), obtained by cutting out the last row and column of (1), equal to zero.

The equation (7) is then homogeneous in $S_1, S_2, S_3, S_4$; by a well-known theorem in determinants the left of (4) is a perfect square, and the equation (6) for $1/\rho$ has equal roots. The two circles $\rho$ coincide in this case.

(ii) By putting $\cos\alpha = \cos\beta = \cos\gamma = 0$ in (6) so that the circle $\rho$ is the orthogonal circle, we get

$$k^2 \begin{vmatrix} 0 & 1/a & 1/b & 1/c \\ 1/a & 1 & \cos C & \cos B \\ 1/b & \cos C & 1 & \cos A \\ 1/c & \cos B & \cos A & 1 \end{vmatrix} = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}.$$

The determinant which multiplies $k^2$ is easily shown to be $-4\Delta^2_{def}/a^2b^2c^2$.

If the radii and $\Delta_{def}$ are finite the circles will have a common point if

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{aligned} &\sin\tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(B+C-A) \\ &\sin\tfrac{1}{2}(C+A-B) \\ &\sin\tfrac{1}{2}(A+B-C) = 0. \end{aligned}$$

This also follows from putting

$$r = 0 \text{ and } S_1 = S_2 = S_3 = 0 \text{ in (3).}$$

If in addition $\Delta = 0$, $k$ will be indeterminate and the circles $a, b, c$ will be coaxal.

If $a, b, c$ are all infinite, then from (6), for example,

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0 \text{ again.}$$

This is the relation connecting the supplements of the angles of a rectilineal triangle when their signs are undetermined (see Art. 21).

(iii) Putting $r=0$ in (3) and taking $(x, y)$ at the centre of $\rho$ we find, after slight manipulation

$$\begin{vmatrix} 1 & \cos C & \cos B & 1/a & X \\ \cos C & 1 & \cos A & 1/b & Y \\ \cos B & \cos A & 1 & 1/c & Z \\ 1/a & 1/b & 1/c & 0 & -2 \\ X & Y & Z & -2 & 0 \end{vmatrix} = 0.$$

This is the identical relation connecting the absolute tri-circular coordinates of a point pair.

(iv) Denote by $\Sigma$ what the determinant of (4) becomes when we put $\cos\alpha$, $\cos\beta$, $\cos\gamma$ all equal to zero. $\Sigma=0$ is then the equation of the orthogonal circle.

Also let S denote the determinant of (4) as it stands.

$$\text{Then}\quad S+\kappa\Sigma = \begin{vmatrix} 1 & \cos C & \cos B & \cos\alpha & X \\ \cos C & 1 & \cos A & \cos\beta & Y \\ \cos B & \cos A & 1 & \cos\gamma & Z \\ \cos\alpha & \cos\beta & \cos\gamma, & 1+\kappa, & 0 \\ X & Y & Z & 0 & 0 \end{vmatrix}.$$

If now we determine $\kappa$ so that the determinant obtained from this by leaving out the last row and column is zero, which is possible unless

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0,$$

then the determinant itself is, by the theorem already cited, the square of a linear function of X, Y, Z, say of $pX+qY+rZ$.

$$\text{Thus}\quad S \equiv -\kappa\Sigma + (pX+qY+rZ)^2.$$

The equation of two inverse circles in tricirculars is thus of the same form as the equation of a conic having double contact with a given conic.

Similarly it may be shown that by adding a certain multiple of the determinant of (iii) to $\Sigma$, we obtain a square of the form $\qquad (p'X+q'Y+r'Z+s)^2.$

This obviously ought to be the case since $\Sigma=0$ represents a single circle.

(v) Referring to (1), suppose that the angles of intersection of 4 and 5 with 1, 2, 3 are given. Then (1) gives a quadratic for cos45. The inverse pairs 4 and 5 will *touch*, that is, one of 4 will touch one of 5, and the other of 4 the other of 5, provided the determinant obtained by putting cos45 equal to 1 in (1) is zero. Modify the determinant so obtained by subtracting from the last column the last but one, and similarly with rows; and we see from (4) that the circles 4 and 5 will touch provided the point pair

$$X : Y : Z = \cos 15 - \cos 14 : \cos 25 - \cos 24 : \cos 35 - \cos 34$$

lies on 4, or, similarly, on 5.

This result will be of use later, and its converse will be proved in next Article.

19. *The point of contact of two touching circles given by their angles of section with a, b, c.*

*Theorem.* If two circles $\rho$, $r$ which touch each other cut a circle $a$ at angles $\alpha$, $\theta_1$, and if $\Sigma_1$ = square of tangent to $a$ from the point of contact of $\rho$, $r$,

then $$(\rho - r)\Sigma_1 = 2ar\rho(\cos\alpha - \cos\theta_1). \quad - \quad - \quad (1)$$

Let P, Q, D be the centres of $\rho$, $r$, $a$; R the point of contact of $\rho$, $r$.

The theorem comes from the well-known relation between the mutual distances of D and the three collinear points P, Q, R, viz.,

$$QR . PD^2 + RP . QD^2 + PQ . RD^2 + QR . RP . PQ = 0.$$

It follows that

$$QR(PD^2 - a^2) + RP(QD^2 - a^2) + PQ(RD^2 - a^2) + QR . RP . PQ = 0. \quad (2)$$

But here $$QR = r,\ RP = -\rho,\ PQ = \rho - r;$$
$$PD^2 - a^2 = \rho^2 - 2a\rho\cos\alpha,$$
$$QD^2 - a^2 = r^2 - 2ar\cos\theta_1,$$
$$RD^2 - a^2 = \Sigma_1.$$

Substitute these values in (2) and the theorem follows at once.

20. *An inverse pair of circles as an envelope of orthogonals.*

*Another solution of the problem of section at given angles.*

If $$S_4 \equiv (\lambda S_1/a + \mu S_2/b + \nu S_3/c)/(\lambda/a + \mu/b + \nu/c)$$

the circle $S_4 = 0$ is an orthogonal; let its radius be $r$, and its angles with $a$, $b$, $c$ be $\theta_1$, $\theta_2$, $\theta_3$.

Let any fifth circle R cut $S_1$, $S_2$, $S_3$, $S_4$ at angles $\alpha'$, $\beta'$, $\gamma'$, $\delta$.

Then for the values of $S_1$, $S_2$, $S_3$, $S_4$ at the centre of R we have

$$S_1 = R^2 - 2aR\cos\alpha',$$
$$S_2 = R^2 - 2bR\cos\beta',$$
$$S_3 = R^2 - 2cR\cos\gamma',$$
$$S_4 = R^2 - 2rR\cos\delta.$$

Multiply these equations by $\lambda/a$, $\mu/b$, $\nu/c$, $-(\lambda/a + \mu/b + \nu/c)$ and add.

$$\therefore \quad (\lambda/a + \mu/b + \nu/c)r\cos\delta = \lambda\cos\alpha' + \mu\cos\beta' + \nu\cos\gamma'. \quad - \quad (1)$$

For the circle R take in turn the circles $a$, $b$, $c$, $r$.

$$\therefore \quad \left.\begin{aligned} (\lambda/a + \mu/b + \nu/c)r\cos\theta_1 &= \lambda + \mu\cos C + \nu\cos B \\ (\quad \ldots \quad)r\cos\theta_2 &= \lambda\cos C + \mu + \nu\cos A \\ (\quad \ldots \quad)r\cos\theta_3 &= \lambda\cos B + \mu\cos A + \nu \end{aligned}\right\} \quad - \quad - \quad (2)$$

$$\text{and} \quad (\quad \ldots \quad)r = \lambda\cos\theta_1 + \mu\cos\theta_2 + \nu\cos\theta_3. \quad - \quad (3)$$

By elimination of $\theta_1$, $\theta_2$, $\theta_3$ from (2), (3)

$$(\lambda/a + \mu/b + \nu/c)^2r^2 = \lambda^2 + \mu^2 + \nu^2 + 2\mu\nu\cos A + 2\nu\lambda\cos B + 2\lambda\mu\cos C. \quad (4)$$

Finally in (1) for R take a circle $\rho$ which cuts $a$, $b$, $c$ at angles $\alpha$, $\beta$, $\gamma$ and suppose that $S_4$ touches $\rho$.

$$\therefore \quad (\lambda/a + \mu/b + \nu/c)r = \lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma. \quad - \quad - \quad (5)$$

From (4) and (5) it follows that the circle $\rho$ will touch $S_4$ (of radius $r$ determined by (5)) provided

$$\lambda^2+\mu^2+\nu^2+2\mu\nu\cos A+2\nu\lambda\cos B+2\lambda\mu\cos C = (\lambda\cos\alpha+\mu\cos\beta+\nu\cos\gamma)^2. \quad (6)$$

Hence the inverse pair of circles cutting $a$, $b$, $c$ at $\alpha$, $\beta$, $\gamma$ may be considered as the envelope of the orthogonal $\lambda X + \mu Y + \nu Z = 0$, subject to (6).

In the system of coordinates X, Y, Z, (6) is the *tangential equation* of the inverse pair of circles.

It would be easy to show that the condition (6) is equivalent to this, that the radius of the orthogonal is proportional to the $\perp^r$ from its centre on the radical axis of the inverse pair.

As to the *form* of (6), compare 18 (iv).

The equation of the two circles $\rho$ might now be found by following precisely the lines of the method by which, in Conics, the trilinear equation of a conic is deduced from its tangential equation. But

we may use the theorem of 19, which gives, at the point of contact of (6) with its envelope

$$\frac{\rho - r}{2\rho}\frac{S_1}{a} = r\cos a - r\cos\theta_1$$

or multiplying by $\lambda/a + \mu/b + \nu/c$, writing X for $S_1/a$, and using the first of (2),

$$\left.\begin{array}{lll} (\lambda/a+\mu/b+\nu/c)(\rho-r)X/2\rho = (\lambda/a+\mu/b+\nu/c)r\cos a - \lambda - \mu\cos C - \nu\cos B \\ \text{Similarly} \\ (\qquad\qquad)(\quad)Y/2\rho = (\qquad\qquad)r\cos\beta - \lambda\cos C - \mu - \nu\cos A \\ (\qquad\qquad)(\quad)Z/2\rho = (\qquad\qquad)r\cos\gamma - \lambda\cos B - \mu\cos A - \nu \end{array}\right\} . \quad (7)$$

$$\left.\begin{array}{ll} \text{Also from (5)} \quad 0 = (\qquad\qquad)r - \lambda\cos a - \mu\cos\beta - \nu\cos\gamma \\ \text{and we have of course} \quad 0 = \lambda X + \mu Y + \nu Z \end{array}\right\} . \quad (8)$$

By linear elimination of

$$(\lambda/a + \mu/b + \nu/c)(\rho - r)/2\rho,\ (\lambda/a + \mu/b + \nu/c)r,\ \lambda,\ \mu,\ \nu$$

we find 17 (4), the equation of the two circles $\rho$.

To get the equation of the tangent orthogonal at (X′, Y′, Z′) write X′, Y′, Z′ for X, Y, Z in (7) and then eliminate between (7) and (8). The equation is 17 (4) but with X, Y, Z accented either in last column or last row.

If instead of the last of (8) we take the obvious identity

$$(\lambda/a + \mu/b + \nu/c)(\rho - r)(-2)/2\rho = (\lambda/a + \mu/b + \nu/c)r/\rho - \lambda/a - \mu/b - \nu/c$$

and eliminate as before, we obtain 17 (7) with $S_4 = 0$.

Lastly, to get the equation for the radii 17 (6). Put $S_4 = 0$ in 17 (7) and let the minors of the constituents of the last row be L, M, N, P, Q.

Then the determinant

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos a & 1/a \\ \cos C & 1 & \cos A & \cos\beta & 1/b \\ \cos B & \cos A & 1 & \cos\gamma & 1/c \\ \cos a & \cos\beta & \cos\gamma & 1 & 1/\rho \\ S_1/a & S_2/b & S_3/c & 0 & -2 \end{vmatrix} \quad (9) \quad \text{is equal to} \quad LS_1/a + MS_2/b + NS_3/c - 2Q.$$

Now the value of any S at the centre of $S = 0$ is $-$square of radius.

In particular the value of the determinant (9) at the centre of $\rho$ is $-(L/a + M/b + N/c)\rho^2$.

But since, at the centre of $\rho$, $S_1 = \rho^2 - 2a\rho\cos\alpha$ and so on, it follows that the value of the determinant (9) at the centre of $\rho$ is

$$(\rho^2 - 2a\rho\cos\alpha)L/a + (\rho^2 - 2b\rho\cos\beta)M/b + (\rho^2 - 2c\rho\cos\gamma)N/c - 2Q.$$

Equating the two values, we have

$$L\left(\frac{1}{a} - \frac{1}{\rho}\cos\alpha\right) + M\left(\frac{1}{b} - \frac{1}{\rho}\cos\beta\right) + N\left(\frac{1}{c} - \frac{1}{\rho}\cos\gamma\right) - Q \cdot \frac{1}{\rho^2} = 0 \quad (10)$$

which is a determinant obtained from (9) by replacing the last row by the coefficients of L, M, N, P, Q in (10). 17 (6) follows by adding to the last row $(1/\rho)$. last but one.

21. *Method of finding corresponding results for a rectilineal triangle. The circumscribing circles.*

It might be interesting to discuss in detail the limiting forms of the equations of Art. 17 when one or more of the given circles become right lines, but we shall merely notice here the form which equation (4) takes for the case of a rectilineal triangle.

Suppose we have three real intersecting circles with positive radii $a$, $b$, $c$. Keep three of the points of intersection A, B, C fixed and let the centre of $a$ pass to infinity on the side of BC remote from A, and similarly with the other two. It is geometrically obvious that in the limit $S_1/2a$ or $\frac{1}{2}X$ becomes the $\perp^r$ $a$, or $x$ let us say, from the variable point to the line BC. Hence we have only to write $x$, $y$, $z$ for X, Y, Z in 17 (4) to get the equation in trilinears of the circle cutting BC, CA, AB at angles $\alpha$, $\beta$, $\gamma$.

It is essential to notice, however, that the angles A, B, C of the circles become, not the angles A, B, C of the triangle, but their ***supplements.***

Returning to the circles $a$, $b$, $c$, 17 (5) is

$$X^2\{(\cos\beta\cos\gamma - \cos A)^2 - \sin^2\beta\sin^2\gamma\} + \ldots + \ldots$$
$$- 2YZ\{\sin^2\alpha(\cos\beta\cos\gamma - \cos A)$$
$$+ (\cos\alpha\cos\beta - \cos C)(\cos\alpha\cos\gamma - \cos B)\} - \ldots - \ldots = 0. \quad (1)$$

If we choose $\alpha$, $\beta$, $\gamma$ so that the coefficients of $X^2$, $Y^2$ and $Z^2$ in this equation are all zero, the equation will be satisfied if any two of X, Y, Z are zero, and will therefore represent a pair of circles one or other of which passes through each of the six points of intersection of $a$, $b$, $c$. If $b$, $c$ intersect at A, A′; $c$, $a$ at B, B′; and

$a$, $b$ at C, C′, then A, A′ are inverse points as also B, B′ and C, C′. There will clearly be four inverse pairs of *circumscribing circles*, viz.,

ABC, A′B′C′; ABC′, A′B′C; AB′C, A′BC′; AB′C′, A′BC.

To determine $\alpha$, $\beta$, $\gamma$ we have

$$\left.\begin{array}{l}\cos(\beta\pm\gamma)=\cos A\\ \cos(\gamma\pm\alpha)=\cos B\\ \cos(\alpha\pm\beta)=\cos C\end{array}\right\}.$$

As explained more fully in connection with equation (6) of next Article, we reject certain of these solutions as involving that the three circles have a common point.

Further, since it is the *cosines* only of $\alpha$, $\beta$, $\gamma$ that it is material to know, it is easy to see that all the solutions are in effect included in the four

$$\left.\begin{array}{l}\beta+\gamma=A\\ \gamma+\alpha=B\\ \alpha+\beta=C\end{array}\right\}(\text{i});\quad \left.\begin{array}{l}\beta+\gamma=-A\\ \gamma+\alpha=B\\ \alpha+\beta=C\end{array}\right\}(\text{ii});\quad \left.\begin{array}{l}\beta+\gamma=A\\ \gamma+\alpha=-B\\ \alpha+\beta=C\end{array}\right\}(\text{iii});\quad \left.\begin{array}{l}\beta+\gamma=A\\ \gamma+\alpha=B\\ \alpha+\beta=-C\end{array}\right\}(\text{iv}).$$

In (1) the coefficient of $-2YZ$ is

$$\sin^2\alpha(\cos\beta\cos\gamma-\cos A)+(\cos\alpha\cos\beta-\cos C)(\cos\alpha\cos\gamma-\cos B)$$

which, when $\cos(\beta+\gamma)$ is put for $\cos A$, and so on, becomes

$$2\sin^2\alpha\sin\beta\sin\gamma.$$

Hence the equation of any one of the four circumscribing pairs has the form

$$\frac{\sin\alpha}{X}+\frac{\sin\beta}{Y}+\frac{\sin\gamma}{Z}=0.$$

For (i) $\qquad \alpha=M-A,\ \beta=M-B,\ \gamma=M-C,$

where $\qquad M=\frac{1}{2}(A+B+C),$

and the equation is

$$\frac{\sin(M-A)}{X}+\frac{\sin(M-B)}{Y}+\frac{\sin(M-C)}{Z}=0.$$

For the others we have to change the signs of A, B, C respectively both in the expressions for the angles of intersection and in the equation of the pair of circles. In the case of 3 real intersecting circles, the values of the angles of intersection can easily be verified from a figure.

For a rectilineal triangle write $\pi-A$, $\pi-B$, $\pi-C$ for A, B, C.

The equation of the pair (i) becomes

$$\frac{\sin A}{x}+\frac{\sin B}{y}+\frac{\sin C}{z}=0,$$

the well-known equation of the circumscribing circle.

The equation of the second circumscribing pair is

$$\sin\tfrac{1}{2}(A+B+C)/X+\sin\tfrac{1}{2}(C-A-B)/Y+\sin\tfrac{1}{2}(B-A-C)/Z=0.$$

This becomes for a triangle

$$x(y\sin B+z\sin C)=0$$

which represents the base BC and the parallel to it through A.

We find in the same way the equation of the inverse pair of circles with respect to which $a$, $b$, $c$ are *self-polar* (in a sense which we do not explain at present), namely,

$$\frac{X^2\cos A}{\cos A-\cos B\cos C}+\frac{Y^2\cos B}{\cos B-\cos C\cos A}+\frac{Z^2\cos C}{\cos C-\cos A\cos B}=0.$$

This becomes for a triangle, $x^2\sin 2A+y^2\sin 2B+z^2\sin 2C=0$, the known equation.

22. *The Hart Circles.*

Besides the Apollonian circles and the circumscribing circles, there are some other sets of four inverse pairs related in an interesting way to the given tri-circle, notably the circles discovered by Hart, each pair of which touches each Apollonian pair. We shall investigate these by determining their angles of intersection with $a$, $b$, $c$.

The condition, 16 (3), that four circles should be co-tangible by a circle is, in terms of the common tangents,

$$12\cdot 34\pm 13\cdot 24\pm 14\cdot 23=0, \qquad - \quad - \quad - \quad (1)$$

or, in terms of the angles between the circles,

$$\sin\tfrac{1}{2}12\cdot\sin\tfrac{1}{2}34\pm\sin\tfrac{1}{2}13\cdot\sin\tfrac{1}{2}24\pm\sin\tfrac{1}{2}14\cdot\sin\tfrac{1}{2}23=0. \quad - \quad (2)$$

We propose to enquire whether a circle $\rho$, cutting $a$, $b$, $c$ at $\alpha$, $\beta$, $\gamma$ can be found such that each of the four sets of circles

$$\rho,\ a,\ b,\ c\ ;\quad -\rho,\ -a,\ b,\ c\ ;\quad -\rho,\ a,\ -b,\ c\ ;\quad -\rho,\ a,\ b,\ -c \quad - \quad (3)$$

is a cotangible set.

$\rho$, $a$, $b$, $c$ will be cotangible if

$$\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}A\pm\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}B\pm\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}C=0.$$

The angles between $-a$ and $b$, $c$, $-\rho$ are $\pi-C$, $\pi-B$, $a$ respectively, and therefore $-\rho$, $-a$, $b$, $c$ will be cotangible if

$$\sin\tfrac{1}{2}a\sin\tfrac{1}{2}A\pm\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}B\pm\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}C=0.$$

Each of the sets (3) will therefore be cotangible, provided we can find $a$, $\beta$, $\gamma$ so that

$$\left.\begin{aligned} \sin\tfrac{1}{2}a\sin\tfrac{1}{2}A+\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}C=0\\ \sin\tfrac{1}{2}a\sin\tfrac{1}{2}A+\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}B-\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}C=0\\ -\cos\tfrac{1}{2}a\cos\tfrac{1}{2}A+\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}B+\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}C=0\\ \cos\tfrac{1}{2}a\cos\tfrac{1}{2}A-\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}B+\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}C=0 \end{aligned}\right\} \quad . \quad . \quad (4)$$

There are here only three independent equations, as the first is found on adding the other three. This would happen with various other distributions of the signs of the terms, but it will be found that the altered equations are really the same as (4), with different initial determinations of $a$, $\beta$, $\gamma$ from their cosines. (2) ($j$)).

The equations (4) are equivalent to

$$\left.\begin{aligned} \cos\tfrac{1}{2}(\beta+B)-\cos\tfrac{1}{2}(\gamma-C)=0\\ \cos\tfrac{1}{2}(\gamma+C)-\cos\tfrac{1}{2}(a-A)=0\\ \cos\tfrac{1}{2}(a+A)-\cos\tfrac{1}{2}(\beta-B)=0 \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

or to

$$\left.\begin{aligned} \beta+B=4n_1\pi\pm(\gamma-C)\\ \gamma+C=4n_2\pi\pm(a-A)\\ a+A=4n_3\pi\pm(\beta-B) \end{aligned}\right\} \quad (6) \quad \text{where the } n\text{'s are integers.}$$

If we take all the ambiguous signs in (6) + we get by addition $A+B+C=2N\pi$; if we take the first ambiguity + and the other two $-$ we get $B+C-A=2N'\pi$.

But if $A\pm B\pm C=2M\pi$ the circles $a$, $b$, $c$ have a common point, (18 (ii)).

Neglecting consideration of this special case, we have to deal with four sets of equations, viz.,

$$\left.\begin{aligned} \beta+B=4n_1\pi-\gamma+C\\ \gamma+C=4n_2\pi-a+A\\ a+A=4n_3\pi-\beta+B \end{aligned}\right\} \quad \text{and three sets of the type} \quad \left.\begin{aligned} \beta+B=4n_1\pi-\gamma+C\\ \gamma+C=4n_2\pi+a-A\\ a+A=4n_3\pi+\beta-B \end{aligned}\right\}.$$

Now it makes no difference to the determination of the circles $a$, $\beta$, $\gamma$ if the signs of any or all of $a$, $\beta$, $\gamma$ are changed, or if any multiple of $2\pi$ is added to any of them.

Hence we may take as the four solutions

| | $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|---|
| $\alpha$ | $B-C$ | $B-C$ | $B+C$ | $B+C$ |
| $\beta$ | $C-A$ | $C+A$ | $C-A$ | $C+A$ |
| $\gamma$ | $A-B$ | $A+B$ | $A+B$ | $A-B$ |

- - (7)

and we have proved that each of the eight circles thus specified, with the sign of its radius determined as in the scheme (3), touches four Apollonian circles, one from each inverse pair. Obviously when a Hart circle touches four Apollonian circles, its inverse touches the other four. From this it follows easily that each Apollonian circle touches four Hart circles, one from each pair.

For, denote the Appollonian pairs by $A_0$, $A_1$, $A_2$, $A_3$; $A_0$ touching $a$, $b$, $c$; $A_1$ touching $-a$, $b$, $c$, etc., and let their radii be $a_0$, $a_0'$; $a_1$, $a_1'$, etc. Also let $h_0$, $h_0'$; $h_1$, $h_1'$, etc., be the radii of the Hart circles. Each *pair* of radii is perfectly definite, being the roots of 17 (6) with the proper values of $\alpha$, $\beta$, $\gamma$.

Now since $h_0$ is cotangible with $a$, $b$, $c$ it touches either $a_0$ or $a_0'$; if it is $a_0'$, then $h_0'$, $a_0$, being the inverses of $h_0$, $a_0'$, touch. Hence either $h_0$ or $h_0'$ touches $a_0$. Similarly every Apollonian circle is touched by one from each Hart pair.

We can indicate the nature of the different contacts more definitely.

Thus, attaching the symbol $a_0$ arbitrarily to one of the pair of radii of $A_0$, denote by $h_0$, $h_1$, $h_2$, $h_3$ the radii of the circles which $a_0$ touches; then by $a_1$, $a_2$, $a_3$ the radii of the circles which $-h_0$ touches.

Then $a_0$ touches $h_0$, $h_1$, $h_2$, $h_3$

and $\therefore$ $a_0'$ ...... $h_0'$, $h_1'$, $h_2'$, $h_3'$.

Also $-h_0'$ ...... $a_1'$, $a_2'$, $a_3'$.

Next $a_1$ touches either $-h_1$ or $-h_1'$; *the question is, which?* The following method not only answers this question, but it gives an interesting expression for the angle between $h_0$ and $h_1$. (The *two* possible values could be found from 17 (1)).

We know that $a$, $b$, $c$, $h_0$, $h_1$ are cotangible; hence from (2),

$$\left.\begin{aligned}
&\sin\tfrac{1}{2}A\sin\tfrac{1}{2}h_0h_1 \pm \sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A+B) \pm \sin\tfrac{1}{2}(C+A)\sin\tfrac{1}{2}(A-B)=0,\\
&\sin\tfrac{1}{2}B\sin\tfrac{1}{2}h_0h_1 \pm \sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(B-C) \pm \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B-C)=0,\\
&\sin\tfrac{1}{2}C\sin\tfrac{1}{2}h_0h_1 \pm \sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C+A) \pm \sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)=0,
\end{aligned}\right\}($$

The common solution of these equations is

$$\sin\tfrac{1}{2}h_0h_1 = 2\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}A, \qquad (9)$$

(strictly, $\pm$ this, but the sign is immaterial.)

In the first of the identities (8)

change B into $\pi + B$, and C into $\pi + C$.

This gives

$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}h_0h_1 \pm \cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A+B) \mp \cos\tfrac{1}{2}(C+A)\cos\tfrac{1}{2}(A-B) = 0,$$

which is just the condition that $b$, $c$, $-h_0$, $-h_1$ should be cotangible.

Similarly we show that $c$, $-a$, $-h_0$, $-h_1$ and $-a$, $b$, $-h_0$, $-h_1$ are cotangible.

Hence $-a$, $b$, $c$, $-h_0$, $-h_1$ are cotangible.

We can prove in the same way that

$$\left.\begin{aligned} \sin\tfrac{1}{2}h_0h_1' &= 2\cos\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}A \\ \sin\tfrac{1}{2}h_2h_3 &= 2\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}A \\ \sin\tfrac{1}{2}h_2h_3' &= 2\cos\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}A \end{aligned}\right\} \qquad (10)$$

and the rest can be written down from symmetry.

We have then the following scheme of contacts

$$\begin{array}{lllllllll}
\begin{cases} a_0 \\ a_0' \end{cases} & \begin{matrix}\text{touches} \\ \ \end{matrix} & \begin{matrix} a, & b, & c, & h_0, & h_1, & h_2, & h_3 \\ a, & b, & c, & h_0', & h_1', & h_2', & h_3' \end{matrix} \\[2ex]
\begin{cases} a_1 \\ a_1' \end{cases} & & \begin{matrix} -a, & b, & c, & -h_0, & -h_1, & -h_2', & -h_3' \\ -a, & b, & c, & -h_0', & -h_1', & -h_2, & -h_3 \end{matrix} \\[2ex]
\begin{cases} a_2 \\ a_2' \end{cases} & & \begin{matrix} a, & -b, & c, & -h_0, & -h_1', & -h_2, & -h_3' \\ a, & -b, & c, & -h_0', & -h_1, & -h_2', & -h_3 \end{matrix} \\[2ex]
\begin{cases} a_3 \\ a_3' \end{cases} & & \begin{matrix} a, & b, & -c, & -h_0, & -h_1', & -h_2', & -h_3 \\ a, & b, & -c, & -h_0', & -h_1, & -h_2, & -h_3'. \end{matrix}
\end{array} \qquad (11)$$

From the scheme we see that the eight Apollonian circles can be arranged in sets of four, one from each pair, in eight ways, so that the four circles of each set are touched by another circle besides

$\pm a, \pm b, \pm c$, viz., by $h_0$ or $h_0'$ or $h_1$, etc.

$$\left.\begin{array}{llllll}
\text{For instance} & -a_0, & a_1, & a_2, & a_3 \text{ are touched by} & -h_0 \\
& a_0, & -a_1, & a_2, & a_3 \quad \ldots\ldots\ldots & a \\
& a_0, & a_1, & -a_2, & a_3 \quad \ldots\ldots\ldots & b \\
& a_0, & a_1, & a_2, & -a_3 \quad \ldots\ldots\ldots & c
\end{array}\right\}. \qquad (12)$$

Any three of the Apollonian circles fall into one of these sets of four, but the fourth is determined when the three are chosen.

Further, when three are chosen, the four circles which touch them belong, one to each of the Apollonian pairs of the three.

We can thus specify all eight Apollonian circles of any three of the Apollonian circles of $\pm a$, $\pm b$, $\pm c$. Thus, taking $\pm a_1$, $\pm a_2$, $\pm a_3$ we see from (12) that $-h_0$, $a$, $b$, $c$ are four of their Apollonian circles, one from each pair, and the other four are therefore the inverses of these with respect to the orthogonal circle of $a_1$, $a_2$, $a_3$.

23. The equation of any Hart pair is found from 21 (1), viz.,

$$X^2\{(\cos\beta\cos\gamma - \cos A)^2 - \sin^2\beta\sin^2\gamma\} + \ldots + \ldots$$

$$-2YZ\{\sin^2\alpha(\cos\beta\cos\gamma - \cos A)$$

$$+ (\cos\alpha\cos\beta - \cos C)(\cos\alpha\cos\gamma - \cos B)\} - \ldots - \ldots = 0$$

by substituting the values of $\alpha$, $\beta$, $\gamma$ from 22 (7).

For $H_0$, the coefficient of $X^2$ is

$$(\cos\overline{\beta - \gamma} - \cos A)(\cos\overline{\beta + \gamma} - \cos A)$$

$$= 4\sin\frac{3A - B - C}{2}\sin\frac{B + C - A}{2}\sin\frac{C + A - B}{2}\sin\frac{A + B - C}{2};$$

the coefficient of $-2YZ$ is

$$\cos\beta\cos\gamma - \cos A + \cos\alpha(\cos A\cos\alpha - \cos B\cos\beta - \cos C\cos\gamma) + \cos B\cos C$$

$$= \cos\beta\cos\gamma - \cos A - \cos\alpha\cos(B + C - A) + \cos B\cos C$$

$$= (\cos\overline{B - C} - \cos A)(1 - \cos\overline{B + C - A})$$

$$= 4\sin^2\frac{B + C - A}{2}\sin\frac{C + A - B}{2}\sin\frac{A + B - C}{2}.$$

Hence the equation of $H_0$ is

$$X^2\sin\frac{B+C-3A}{2} + Y^2\sin\frac{C+A-3B}{2} + Z^2\sin\frac{A+B-3C}{2}$$

$$+ 2YZ\sin\frac{B+C-A}{2} + 2ZX\sin\frac{C+A-B}{2} + 2XY\sin\frac{A+B-C}{2} = 0. \quad (1)$$

To get the equations of $H_1$, $H_2$, $H_3$ it is clear from 22 (7) that we have only to change the signs of A, B, C respectively in this.

When the circles become right lines, (1) becomes, as in Art. 21,

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C - 2yz\sin A - 2zx\sin B - 2xy\sin C = 0$$

which is the known equation of the nine-points circle.

Again, the equation of $H_1$ is

$$X^2\sin\frac{B+C+3A}{2} + Y^2\sin\frac{C-A-3B}{2} + Z^2\sin\frac{B-A-3C}{2}$$

$$+ 2YZ\sin\frac{A+B+C}{2} + 2ZX\sin\frac{C-A-B}{2} + 2XY\sin\frac{B-A-C}{2} = 0 \qquad (2)$$

which degenerates into

$$x^2\sin A + y^2\sin(B-C) - z^2\sin(B-C) + 2zx\sin C + 2xy\sin B = 0$$

or
$$(x\sin A + y\sin B + z\sin C)^2 - (y\sin C + z\sin B)^2 = 0,$$

which represents two parallel lines, which are easily seen to be, one the fourth common tangent to the inscribed and the first escribed circle, the other the fourth common tangent to the other two escribed circles.

24. The absolute tricircular coordinates of the point of contact of any Hart circle with an Apollonian circle which touches it are given by the theorem of Art. 19. This enables us to distinguish the whole 32 points of contact. For example, where $a_0$, $h_0$ touch we have

$$\frac{X}{\sin^2\frac{1}{2}(B-C)} = \frac{Y}{\sin^2\frac{1}{2}(C-A)} = \frac{Z}{\sin^2\frac{1}{2}(A-B)} = \frac{4a_0h_0}{a_0-h_0}. \qquad (1)$$

This gives for an ordinary triangle at the point of contact of the nine-point and the inscribed circle (since X becomes $2x$, etc.)

$$\frac{x}{\sin^2\frac{1}{2}(B-C)} = \frac{y}{\sin^2\frac{1}{2}(C-A)} = \frac{z}{\sin^2\frac{1}{2}(A-B)} = \frac{2(-r)(-\frac{1}{2}R)}{-r+\frac{1}{2}R} \qquad (2)$$

the radii being both negative, when the circles proceed to their limiting forms in such a way as to make X reduce to $+2x$, etc. (Art. 21).

The orthogonal which touches $H_0$, $A_0$ at their pair of points of contact might now be found by the very same method as that by which we find the line which touches a conic at a given point, in trilinear coordinates. (The result, in fact, is proved in Art. 20.)

Or we may proceed thus :

let $\lambda X + \mu Y + \nu Z = 0$ be an orthogonal touching $H_0$, $A_0$.

Then by Art. 20 (1), (4), (6),

$$\left.\begin{aligned}(\lambda/a + \mu/b + \nu/c)r &= \lambda\cos(B-C) + \mu\cos(C-A) + \nu\cos(A-B)\\ &= \lambda \qquad\qquad\quad + \mu \qquad\qquad\quad + \nu\\ \text{and}\quad (\lambda+\mu+\nu)^2 &= \lambda^2+\mu^2+\nu^2+2\mu\nu\cos A + 2\nu\lambda\cos B + 2\lambda\mu\cos C\end{aligned}\right\} \quad (3)$$

Hence, to determine $\lambda : \mu : \nu$ we have

$$\left.\begin{aligned}&\lambda\sin^2\tfrac{1}{2}(B-C) + \mu\sin^2\tfrac{1}{2}(C-A) + \nu\sin^2\tfrac{1}{2}(A-B) = 0\\ \text{and}\quad &\mu\nu\sin^2\tfrac{1}{2}A \qquad + \nu\lambda\sin^2\tfrac{1}{2}B \qquad + \lambda\mu\sin^2\tfrac{1}{2}C \qquad = 0\end{aligned}\right\} \quad (4)$$

In general, this method of determining the common tangent orthogonals of two given inverse pairs would give us *two* common tangents. When the inverse pairs touch, these coincide.

The obvious identity

$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(B-C) + \sin\tfrac{1}{2}B\sin\tfrac{1}{2}(C-A) + \sin\tfrac{1}{2}C\sin\tfrac{1}{2}(A-B) = 0$$

will be found useful in eliminating, say, $\nu$ from (4) and finding

$$\left(\lambda\sin\frac{B}{2}\sin\frac{B-C}{2} - \mu\sin\frac{A}{2}\sin\frac{C-A}{2}\right)^2 = 0.$$

Hence $\lambda$, $\mu$, $\nu$ are as $\sin\frac{1}{2}A/\sin\frac{1}{2}(B-C)$, etc., and the equation of the tangent orthogonal is

$$\frac{X\sin\frac{1}{2}A}{\sin\frac{1}{2}(B-C)} + \frac{Y\sin\frac{1}{2}B}{\sin\frac{1}{2}(C-A)} + \frac{Z\sin\frac{1}{2}C}{\sin\frac{1}{2}(A-B)} = 0. \quad (5)$$

25. *The four circumscribing pairs are touched by other four pairs.*

The analogy between the theory of inverse pairs of circles, and the theory of conics having double contact with a given conic (*Salmon, Conic Sections, Chapter on Invariants*) suggests the theorem that, like the four Apollonian pairs, the four circumscribing pairs are touched by other four pairs of circles.

This is from more than one point of view the *reciprocal*, or *polar*, counterpart of the theorem which we have just discussed at some length, as we hope to explain in detail, along with some other matters, in a supplement to this paper. Meanwhile we merely verify that the theorem is true, by the methods which have been given here.

The equations of the circumscribing pairs (Art. 21) are

$$\left.\begin{array}{l} \sin\overline{M-A}/X+\sin\overline{M-B}/Y+\sin\overline{M-C}/Z=0 \\ -\sin M\ /X+\sin\overline{M-C}/Y+\sin\overline{M-B}/Z=0 \\ \sin\overline{M-C}/X-\sin M\quad /Y+\sin\overline{M-A}/Z=0 \\ \sin\overline{M-B}/X+\sin\overline{M-A}/Y-\sin M\quad /Z=0 \end{array}\right\} \quad (1)$$

where $M=\frac{1}{2}(A+B+C)$. The angles of intersection of these with $a$, $b$, $c$ are for the first pair $M-A$, $M-B$, $M-C$; and for the others, these with the signs of A, B, C respectively changed.

Now we have seen in Art. 18 (v) that the pair of circles cutting $a$, $b$, $c$ at angles $\theta_1$, $\theta_2$, $\theta_3$ will touch the pair

$$\sin\alpha/X+\sin\beta/Y+\sin\gamma/Z=0$$

which cut $a$, $b$, $c$ at angles $\alpha$, $\beta$, $\gamma$ if

$$\frac{\sin\alpha}{\cos\alpha-\cos\theta_1}+\frac{\sin\beta}{\cos\beta-\cos\theta_2}+\frac{\sin\gamma}{\cos\gamma-\cos\theta_3}=0. \quad (2)$$

If we have

$$\frac{\sin\alpha}{\cos\alpha+\cos\theta_1}+\frac{\sin\beta}{\cos\beta+\cos\theta_2}+\frac{\sin\gamma}{\cos\gamma+\cos\theta_3}=0 \quad (3)$$

the one pair will still touch the other, provided we take the radii of one of the pairs with their signs changed.

Now we have the identity

$$\frac{\sin(M-A)}{\cos(M-A)-\dfrac{\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A}}+\frac{\sin(M-B)}{\cos(M-B)-\dfrac{\cos\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}B}}$$
$$+\frac{\sin(M-C)}{\cos(M-C)-\dfrac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C}}=0. \quad (4)$$

For the first term on the left is

$$\frac{2\cos\frac{1}{2}A\sin(M-A)}{\cos(\frac{1}{2}B+\frac{1}{2}C-A)-\cos\frac{1}{2}(B-C)}=\frac{\cos\frac{1}{2}A\sin(M-A)}{\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(C-A)},$$

so that the identity proposed is equivalent to

$$\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}A\sin\tfrac{1}{2}(B+C-A)+\text{two similar terms}=0$$

or $\{\sin\frac{1}{2}(A+B-C)-\sin\frac{1}{2}(A-B+C)\}\sin\frac{1}{2}(B+C-A)$

$$+\text{two similar terms}=0,$$

which is obviously true.

In (4) change the signs of A, B, C respectively and we get 3 other identities, say (5), (6), (7).

In (4), (5), (6) and (7) change B into $\pi + B$ and C into $\pi + C$; in the same four equations change C into $\pi + C$ and A into $\pi + A$; and again in the same four A into $\pi + A$ and B into $\pi + B$.

As will be seen at once on writing them down, the 16 identities thus obtained are in virtue of (2) and (3) just the conditions that each of the four pairs (1) with radii properly assigned should touch each of the four pairs of circles whose angles of intersection $\theta_1$, $\theta_2$, $\theta_3$ with $a$, $b$, $c$ are given by the table

$$\begin{array}{ll|l|l|l}
\cos\theta_1 = & \dfrac{\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A} & -\dfrac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A} & \dfrac{\cos\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A} & \dfrac{\sin\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A} \\
\cos\theta_2 = & \dfrac{\cos\frac{1}{2}C\cos\frac{1}{2}A}{\cos\frac{1}{2}B} & \dfrac{\sin\frac{1}{2}C\cos\frac{1}{2}A}{\sin\frac{1}{2}B} & -\dfrac{\sin\frac{1}{2}C\sin\frac{1}{2}A}{\cos\frac{1}{2}B} & \dfrac{\cos\frac{1}{2}C\sin\frac{1}{2}A}{\sin\frac{1}{2}B} . \\
\cos\theta_3 = & \dfrac{\cos\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}C} & \dfrac{\cos\frac{1}{2}A\sin\frac{1}{2}B}{\sin\frac{1}{2}C} & \dfrac{\sin\frac{1}{2}A\cos\frac{1}{2}B}{\sin\frac{1}{2}C} & -\dfrac{\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C}
\end{array}$$

*Seventh Meeting, 8th June* 1906.

JAS. ARCHIBALD, Esq., Vice-President, in the Chair.

## A Symbolic Method in Geometrical Optics.

By EDWARD B. ROSS, M.A.

The formulæ given in Herman's Optics, pages 80, 82, 98, 111, and called Cotes's formulæ, are a little difficult to grasp, and do not lend themselves to manipulation. The notation explained below is useful as a mnemonic. I think it also renders the proofs simpler.

The symbol $\overline{AB}$ for the segment A to B is common; we may modify it slightly by turning the bar into a circumflex. The circumflex can then be split into two parts, an acute and a grave accent; these form convenient marks for the initial and final letters of a segment. $\acute{P}\grave{L}$, $\acute{M}\grave{P}$ are segments, $\grave{P}\acute{L}$ is not a segment.

Products are to be expanded by the distributive law, but the order of the letters must be preserved.

The special rules are only two.

(1) $\acute{P}\grave{Q} = \widehat{PQ}$ the length PQ.

(2) Units and meaningless letters are to be deleted; but if this rule would make a term vanish the value is 1 not 0; just as, in simplifying a fraction we can cancel factors in numerator and denumerator, provided we remember that it will not do if everything goes out to write 0 as the answer.

(In practice the accents may usually be dropped.)

This notation we apply to a system of $n$ coaxial thin lenses at $A_1, \ldots, A_n$, of powers $\kappa_1, \ldots, \kappa_n$. From Q a ray issues making an angle $a_0$ with the axis and strikes the lenses at heights above the axis, $y_1, y_2, \ldots, y_n$; making after these refractions angles $a_1, \ldots, a_n$ with the axis.

Here we may give the formulæ, the brevity and similarity of which are the excuse for this paper.

The standard one is that for the apparent distance. The apparent distance is "the distance from the eye at which the object must be placed to subtend the same angle when viewed directly that it appears to subtend when viewed through the instrument."

Apparent distance of P from Q

$$=\acute{\mathrm{Q}}(1+\grave{\mathrm{A}}_1\kappa_1\acute{\mathrm{A}}_1)(1+\grave{\mathrm{A}}_2\kappa_2\acute{\mathrm{A}}_2)\ldots(1+\grave{\mathrm{A}}_n\kappa_n\acute{\mathrm{A}}_n)\grave{\mathrm{P}}$$

or say $\acute{\mathrm{Q}}\overset{n}{\underset{1}{\Pi}}\grave{\mathrm{P}}$

where $\Pi$ stands for a product of factors of the type $(1+\grave{\mathrm{A}}_r\kappa_r\acute{\mathrm{A}}_r)$ and the affixes are the values of $r$ for the first and last factors.

If the first affix is 1, it will sometimes be dropped.

Angular magnification after lens $n$ of object at $\mathrm{Q}=\acute{\mathrm{Q}}\overset{n}{\underset{1}{\Pi}}$. The power, K, of the system (which is the reciprocal of the focal length) increased by unity is $\overset{n}{\underset{1}{\Pi}}$.

The only apparatus we require for dealing with these symbols is a lemma with two corollaries.

*Lemma.* If $m$ is a number

$$m\acute{\mathrm{R}}\,.\,\grave{\mathrm{A}}_n=m\acute{\mathrm{R}}\,.\,\grave{\mathrm{A}}_{n-1}+m\acute{\mathrm{R}}\,.\,\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n.$$

*Proof.*

$$\begin{aligned} m\acute{\mathrm{R}}\,.\,\grave{\mathrm{A}}_n &= m\widehat{\mathrm{RA}}_n \\ &= m\widehat{\mathrm{RA}}_{n-1} \quad + m\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n \\ &= m\acute{\mathrm{R}}\,.\,\grave{\mathrm{A}}_{n-1}+m\acute{\mathrm{R}}\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n \end{aligned}$$

where we avail ourselves of the second rule to introduce the meaningless letter $\acute{\mathrm{R}}$.

*Cor.* 1. $$\acute{\mathrm{S}}\overset{n-1}{\Pi}\grave{\mathrm{A}}_n=\acute{\mathrm{S}}\overset{n-2}{\Pi}\grave{\mathrm{A}}_{n-1}+\acute{\mathrm{S}}\overset{n-1}{\Pi}\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n.$$

In $\acute{\mathrm{S}}\overset{n-1}{\Pi}$ when expanded, every term ends with an *initial* letter, hence the lemma is applicable.

$$\begin{aligned} \therefore\quad \acute{\mathrm{S}}\overset{n-1}{\Pi}\grave{\mathrm{A}}_n &= \acute{\mathrm{S}}\overset{n-1}{\Pi}\grave{\mathrm{A}}_{n-1}+\acute{\mathrm{S}}\overset{n-1}{\Pi}\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n \\ &= \acute{\mathrm{S}}\overset{n-2}{\Pi}(1+\grave{\mathrm{A}}_{n-1}\kappa_n\acute{\mathrm{A}}_{n-1})\grave{\mathrm{A}}_{n-1}+\acute{\mathrm{S}}\overset{n-1}{\Pi}\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n \\ &= \acute{\mathrm{S}}\overset{n-2}{\Pi}\grave{\mathrm{A}}_{n-1}+\acute{\mathrm{S}}\overset{n-1}{\Pi}\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_n \end{aligned}$$

since $\acute{\mathrm{A}}_{n-1}\grave{\mathrm{A}}_{n-1}=0$.

*Cor.* 2. $$\overset{n-1}{\Pi}\grave{A}_n = \overset{n-2}{\Pi}\grave{A}_{n-1} + (\overset{n-1}{\Pi} - 1)\acute{A}_{n-1}\grave{A}_n.$$

We can apply the lemma here again to every term of $\overset{n-1}{\Pi}\grave{A}_n$ except the one $1.1.\dots 1.\grave{A}_n$ which is 1. If the lemma held it would be represented by $\grave{A}_{n-1} + \acute{A}_{n-1}\grave{A}_n$. $\acute{A}_{n-1}\grave{A}_n$ must therefore be subtracted from the right-hand side.

The optical equations are

$$y_1 = a_0 Q A_1 \quad - \quad - \quad - \quad - \quad - \quad (1)$$

$$a_1 - a_0 = y_1\kappa_1 \quad - \quad - \quad - \quad - \quad - \quad (2)$$

$$\dots\dots\dots$$

$$y_r - y_{r-1} = a_{r-1}A_{r-1}A_r \quad - \quad - \quad - \quad (2r-1)$$

$$a_r - a_{r-1} = y_r\kappa_r \quad - \quad - \quad - \quad (2r)$$

$$\dots\dots\dots\dots\dots\dots$$

The formulæ to be established are

$$\frac{y_r}{a_0} = \acute{Q}\overset{r-1}{\Pi}\grave{A}_r \quad - \quad - \quad - \quad - \quad - \quad \text{I.}$$

$$\frac{a_r}{a_0} = \acute{Q}\overset{r}{\Pi} \quad - \quad - \quad - \quad - \quad - \quad \text{II.}$$

I. gives $\dfrac{y_1}{a_0} = \acute{Q}\grave{A}_1$ which agrees with (1).

II. gives $\dfrac{a_1}{a_0} = \acute{Q}(1 + \grave{A}_1\kappa_1\acute{A}_1) = 1 + \kappa_1 Q A_1$.

Assuming that $\dfrac{y_{r-1}}{a_0} = \acute{Q}\overset{r-2}{\Pi}\grave{A}_{r-1}$ and $\dfrac{a_{r-1}}{a_0} = \acute{Q}\overset{r-1}{\Pi}$:

from $(2r-1)$, $$\frac{y_r}{a_0} = \frac{y_{r-1}}{a_0} + \frac{a_{r-1}}{a_0}A_{r-1}A_r$$

$$= \acute{Q}\overset{r-2}{\Pi}\grave{A}_{r-1} + \acute{Q}\overset{r-1}{\Pi}\acute{A}_{r-1}\grave{A}_r$$

$$= \acute{Q}\overset{r-1}{\Pi}\grave{A}_r \qquad \text{by Cor. 1;}$$

and from $(2r)$ $$\frac{a_r}{a_0} = \frac{a_{r-1}}{a_0} + \kappa_r\frac{y_r}{a_0}$$

$$= \acute{Q}\overset{r-1}{\Pi} + \acute{Q}\overset{r-1}{\Pi}\grave{A}_r\kappa_r$$

$$= \acute{Q}\overset{r}{\Pi}.$$

So the formulæ are established by the induction method.

K is the coefficient of $QA_1$ in $\dfrac{a_n - a_0}{a_0}$

$$i.e., \quad K + 1 = \frac{\partial}{\partial Q}\acute{Q}\overset{n}{\Pi}$$

to use an obvious notation

$$= \overset{n}{\Pi}.$$

The apparent distance from Q of a point P after the last lens might be worked out separately, but a simpler way is to take an extra lens at P.

$$\text{Apparent distance} \quad = \frac{y_{n+1}}{a_0} = \acute{Q}\underset{1}{\overset{n}{\Pi}}\grave{P}.$$

The symmetry of this expression is important.

If P is R, the image of Q, the apparent distance is 0; or, in other words, the height of a ray diverging from Q is zero at P.

$$0 = \acute{Q}\underset{1}{\overset{n}{\Pi}}\grave{R}.$$

To find the linear magnification, shift Q back a little, $\Delta Q$. $\Delta Q \, . \, a_0$ = height of object, $\Delta y_{n+1} = y_{n+1}$ is height of image.

$\therefore \quad \dfrac{\partial}{\partial Q}\left(\acute{Q}\overset{n}{\Pi}\grave{R}\right)$ is the linear magnification, *i.e.*, $\overset{n}{\Pi}\grave{R}$.

Similarly $\acute{Q}\overset{n}{\Pi}$ is the reciprocal.

This is $a_n/_{a_0}$ in accordance with Helmholtz's Theorem.

So we have $\quad \left[\acute{Q}\overset{n}{\Pi}\right]\left[\overset{n}{\Pi}\grave{R}\right] = 1$

where the square bracket means arithmetical value.

This may be written

$$\left[\acute{Q}\grave{A}_1\left(\underset{1}{\overset{n}{\Pi}} - 1\right) + \acute{A}_1\underset{2}{\overset{n}{\Pi}}\right]\left[\underset{1}{\overset{n-1}{\Pi}}\grave{A}_n + \left(\overset{n}{\Pi} - 1\right)\acute{A}_n\grave{R}\right] = 1$$

$$\text{or} \quad \left[QA \, . \, K \;\; + \frac{\partial K}{\partial \kappa_1}\right]\left[\frac{\partial K}{\partial \kappa_n} \;\; + KA_nR\right] = 1.$$

The other equation is

$$\acute{Q}\underset{1}{\overset{n}{\Pi}}\grave{R} = 0$$

$$i.e., \quad \acute{Q}\underset{1}{\overset{n-1}{\Pi}}\grave{A}_n + \acute{Q}\underset{1}{\overset{n}{\Pi}} \, . \, \acute{A}_nR = 0$$

$$\text{or}\quad \acute{Q}\grave{A}_1\prod_1^{n-1} + \acute{A}_1\prod_2^{n-1}\grave{A}_n + \acute{Q}\grave{A}_1\left(\prod_1^{n} - 1\right)\acute{A}_n R$$

$$+ \acute{A}_1\prod_2^{n}\acute{A}_n\grave{R}$$

$$\text{or}\quad K \,.\, QA_1 \,.\, A_1R + QA \,.\, \frac{\partial K}{\partial \kappa_n} + AR \,.\, \frac{\partial K}{\partial \kappa_1} + \frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} = 0.$$

Comparing these equations we derive the identity

$$\frac{\partial K}{\partial \kappa_1} \,.\, \frac{\partial K}{\partial \kappa_n} - K\frac{\partial^2 K}{\partial \kappa_1 \partial \kappa_n} = 1.$$

This we may verify directly. Call the quantity on the left $c_n$.

$$c_n = \left[\prod_1^{n-1}\grave{A}_n\right]\left[\acute{A}_1\prod_2^{n}\right] - \left[\prod_1^{n} - 1\right]\left[\acute{A}_1\prod_2^{n-1}\grave{A}_n\right]$$

(separate $\kappa_n$)

$$= \left[\prod_1^{n-1}\grave{A}_n\right]\left[\acute{A}_1\prod_2^{n}\right] - \left[\prod_1^{n-1} - 1\right]\left[\acute{A}_1\prod_2^{n-1}\grave{A}_n\right]$$

$$+ \left[\prod_1^{n-1}\grave{A}_n\right]\left[\acute{A}_1\prod_2^{n}\grave{A}_n\kappa_n\right] - \left[\prod_1^{n-1}\grave{A}_n\kappa_n\right]\left[\acute{A}\prod_2^{n-1}\grave{A}_n\right]$$

where the second line is zero

$$= \left[\prod_1^{n-2}\grave{A}_{n-1} + \left(\prod_1^{n-1} - 1\right)\acute{A}_{n-1}\grave{A}_n\right]\left[\acute{A}_1\prod_2^{n-1}\right]$$

$$- \left[\prod_1^{n-1} - 1\right]\left[\acute{A}_1\prod_2^{n-2}\grave{A}_{n-1} + \acute{A}_1\prod_2^{n-1}\acute{A}_{n-1}\grave{A}_n\right]$$

by corollaries to lemma,

$$= c_{n-1}.$$

Now $\quad c_2 = (1 + A_1\kappa_1A_1)A_2 \,.\, A_1(1 + A_2\kappa_2A_2)$

$$- [(1 + A_1\kappa_1A_1)(1 + A_2\kappa_2A_2) - 1]A_1A_2$$

$$= (1 + \kappa_1A_1A_2)1 + A_1A_2\kappa_2)$$

$$- (\kappa_1 + \kappa_2 + \kappa_1A_1A_2\kappa_2)A_1A_2$$

$$= 1.$$

So the identity is verified.

Thick lenses may be treated in the usual way by reduced distances.

## Theorems connected with Simson's Line.

By A. G. BURGESS, M.A.

[A digest of Mr Burgess' paper is given below.]

FIGURE 16.

(i) If XYZ is the Simson Line P(ABC); if PM is perpendicular to XYZ and cuts the sides BC, CA, AB in U, V, W:

then $\quad PU \, . \, PV \, . \, PW = PA \, . \, PB \, . \, PC = 4R^2 . \, PM.$

(ii) If $PX_1$, $PY_1$, $PZ_1$; $PX_2$, $PY_2$, $PZ_2$ are the two sets of three straight lines which make angle $\alpha$ with the sides of the triangle ABC; so that $X_1$, $Y_1$, $Z_1$ are collinear, and $X_2$, $Y_2$, $Z_2$; and if $X_1Y_1Z_1$, $X_2Y_2Z_2$, intersect in Q:

Q is shown to lie on PM, and the Simson Line P(ABC) is shown to be the following Simson Lines

$$P(QX_1X_2), \text{ etc.}; \; P(AY_1Z_1), \text{ etc.}; \; P(AY_2Z_2), \text{ etc.}$$

(iii) *When* $\alpha = 45°$,

M is the mid-point of PQ; and if O is the orthocentre of ABC, OQ is therefore parallel to XYZ (since XYZ bisects OP). The locus of Q, as P moves on the circle, is given by the equation

$$\rho = 2R[\cos A \cos\theta - \sin\theta \sin\{2\theta - (B - C)\}]$$

with reference to O as pole and OA as initial line. The curve has three loops of different sizes and can easily be traced from the fact that OQ is at right angles to PQ.

If, in particular, ABC is equilateral (so that O is the circumcentre), the locus of Q is given by $\rho = R\cos 3\theta$, a hypotrochoid with three loops each of which is in area one-twelfth of the circle.

FIGURE 17.

If $PX_1$, $PY_1$, $PZ_1$; $PX_2$, $PY_2$, $PZ_2$ make angle $\alpha_1$ with the sides of ABC;
and $PX_3$, $PY_3$, $PZ_3$; $PX_4$, $PY_4$, $PZ_4$ ,, ,, $\alpha_2$ ,, ,, ,, ,, ,, ;
if $Q_1$, $Q_2$ are the two corresponding positions of Q and if the four lines $X_1Y_1Z_1$, etc., intersect one another besides in $T_1$, $T_2$, $T_3$, $T_4$ and intersect the Simson Line P(ABC) in $L_1$, $L_2$, $L_3$, $L_4$:

(i) $T_1$, $T_2$, $T_3$, $T_4$ lie on a circle which has P as centre and cuts orthogonally the circles $T_1Q_1Q_2$, $T_2Q_1Q_2$, $T_3Q_1Q_2$, $T_4Q_1Q_2$;

(ii) $L_1, L_2, L_3, L_4$ are the mid-points of $T_1T_3$, $T_2T_4$, $T_1T_4$, $T_2T_3$; and $T_1T_2$, $T_3T_4$ are parallel to XYZ and equidistant from it;

(iii) the Simson Line P(ABC) is also the following 34 Simson Lines:—

$P(AY_1Z_1)$, etc.; $P(BZ_1X_1)$, etc.; $P(CX_1Y_1)$, etc.; $P(Q_1X_1X_2)$, $P(Q_2X_3X_4)$, etc., etc.; $P(Q_1T_1T_4)$, $P(Q_1T_2T_3)$, $P(Q_2T_1T_3)$, $P(Q_2T_2T_4)$; $P(T_1X_1X_3)$, $P(T_2X_2X_4)$, $P(T_3X_1X_4)$, $P(T_4X_2X_3)$, etc., etc.

---

If $A_1$, $A_2$, $A_3$, $A_4$ be four concyclic points; $O_1, \ldots, O_4$ the orthocentres of the four triangles formed by them:
the quadrilaterals $A_1A_2A_3A_4$, $O_1O_2O_3O_4$ are equal in all respects, and if C, F be their circumcentres, the four Simson Lines of $A_1$, $A_2$, $A_3$, $A_4$ are concurrent at the mid-point P of CF, which is also the mid-point of $A_1O_1, \ldots, A_4O_4$.

FIGURE 18.

If $A_1, \ldots, A_5$ are points on a circle whose centre is C; $O_1, \ldots, O_5$ the orthocentres of the five triangles formed by sets of three consecutive vertices of the pentagon $A_1A_2A_3A_4A_5$; $Q_1, \ldots, Q_5$ the orthocentres of the five triangles formed each by one side and the opposite vertex; if $B_1, \ldots, B_5$, are the mid-points of the sides, and $G_1, \ldots, G_5$ the mid-points of the diagonals; if $F_1, \ldots, F_5$ are the circumcentres of the five cyclic quadrilaterals (see above) formed by orthocentres of triangles whose vertices are chosen from $A_1, \ldots, A_5$; and $P_1, \ldots, P_5$ are the mid-points of $CF_1, \ldots, CF_5$:

It is clear that the pentagon $P_1P_2P_3P_4P_5$ has its sides parallel to and half the length of the sides of the original pentagon; and if D is the circumcentre of this cyclic pentagon, and S the point of trisection such that $CS = 2SD$, the five straight lines $A_1P_1, \ldots, A_5P_5$ are concurrent at S which is a point of trisection of each, as also of the other ten lines $O_1B_1, \ldots, O_5B_5$, $Q_1G_1, \ldots, Q_5G_5$; S being the centre of homology of the two similar and similarly-situated pentagons.

Again the pentagon $F_1F_2F_3F_4F_5$ is equal in all respects to $A_1A_2A_3A_4A_5$ and similarly situated to it; and if E be its circumcentre, D bisects CE and is the centre of homology of $A_1A_2A_3A_4A_5$, $F_1F_2F_3F_4F_5$.

The theory can be extended according to the following table:—

| Number of given points $A_1, A_2, \ldots$ | Specification of polygon. | Ratio of linear dimensions to original polygon. | Circumcentre denoted by | Centre of homology with original polygon. |
|---|---|---|---|---|
| 4 | $O_1O_2O_3O_4$ | 1 | F | P |
| 5 | $F_1F_2F_3F_4F_5$ | 1 | E | D |
| | $P_1P_2P_3P_4P_5$ | $\frac{1}{2}$ | D | S |
| 6 | $E_1E_2E_3E_4E_5E_6$ | 1 | H | K |
| | $D_1D_2D_3D_4D_5D_6$ | $\frac{1}{2}$ | K | L |
| | $S_1S_2S_3S_4S_5S_6$ | $\frac{1}{3}$ | L | M |
| 7 | $H_1H_2H_3H_4H_5H_6H_7$ | 1 | etc. | etc. |
| | $K_1$ ............... $K_7$ | $\frac{1}{2}$ | | |
| | $L_1$ . ............. $L_7$ | $\frac{1}{3}$ | | |
| | $M_1$ ............... $M_7$ | $\frac{1}{4}$ | | |
| | | etc. | | |

# Edinburgh Mathematical Society.

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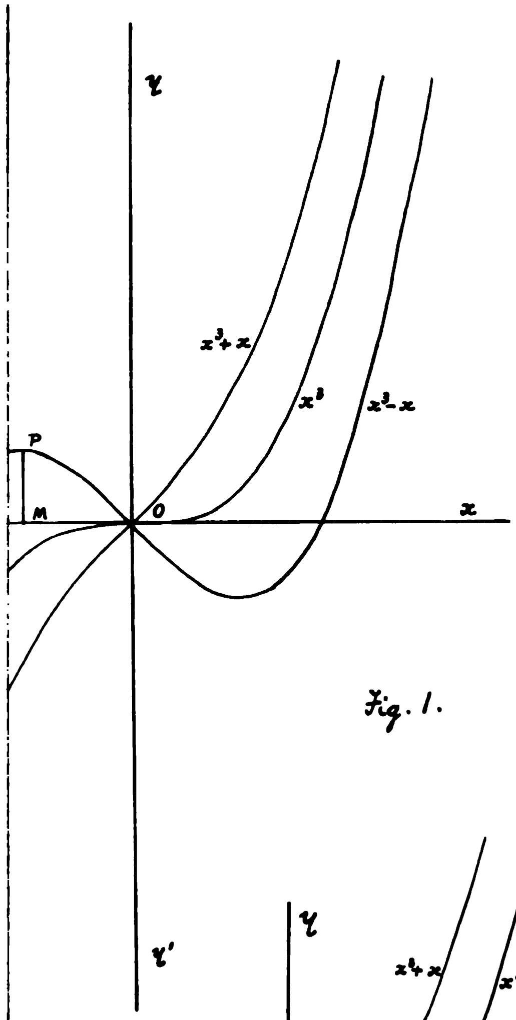


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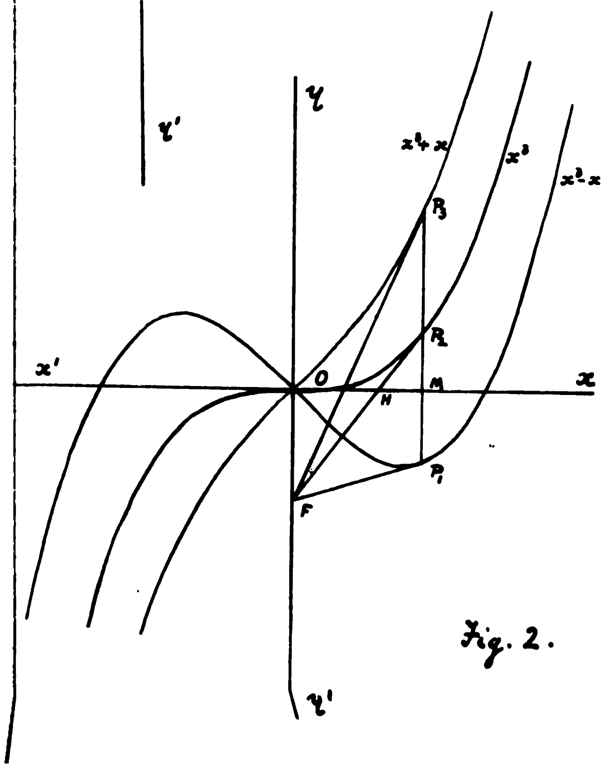


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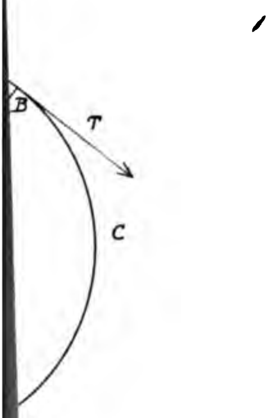


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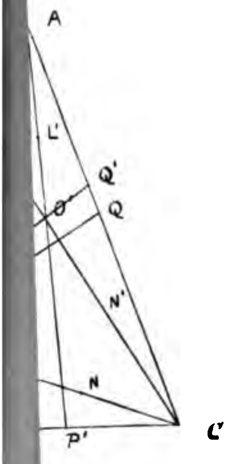

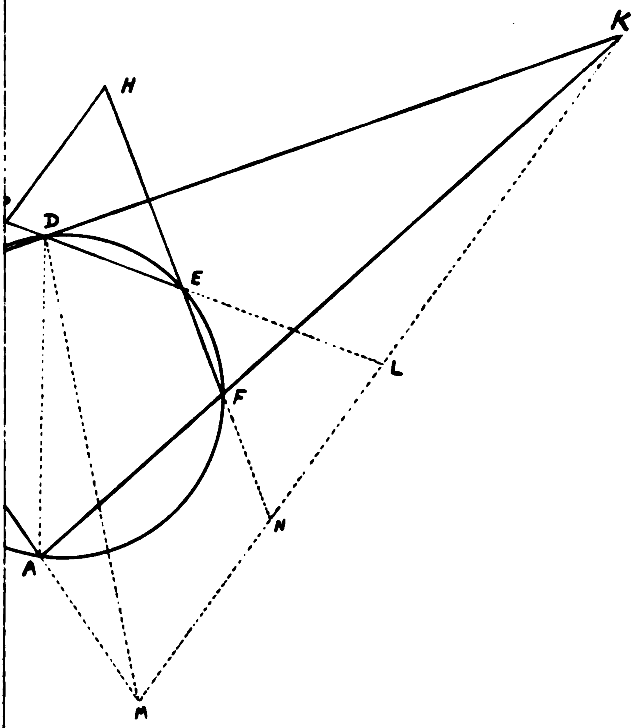


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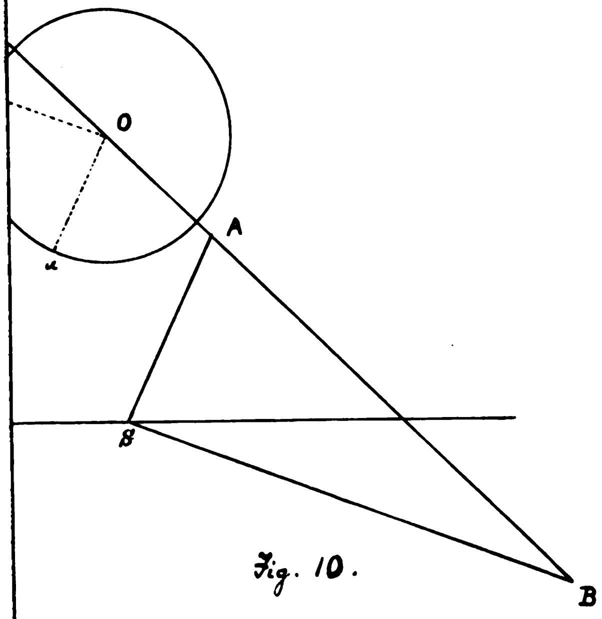


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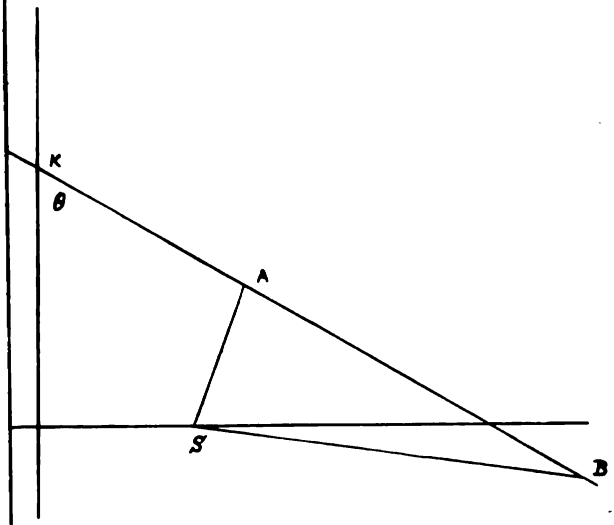


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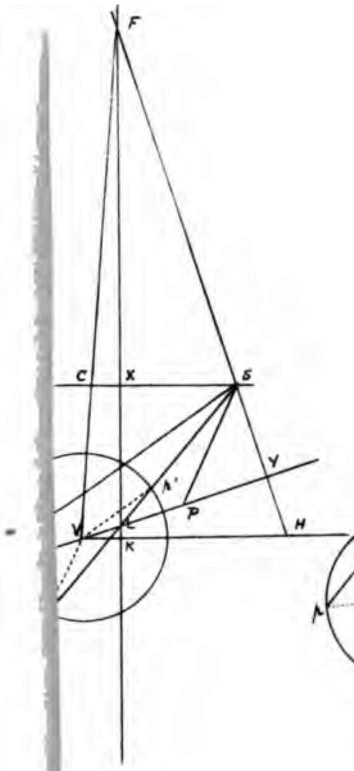


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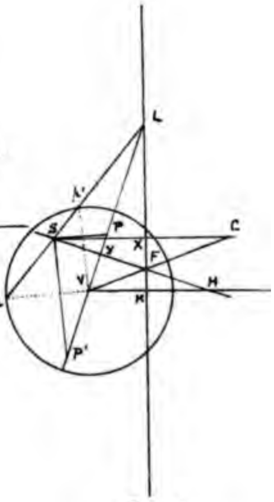


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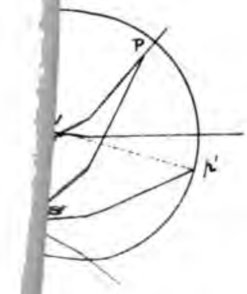


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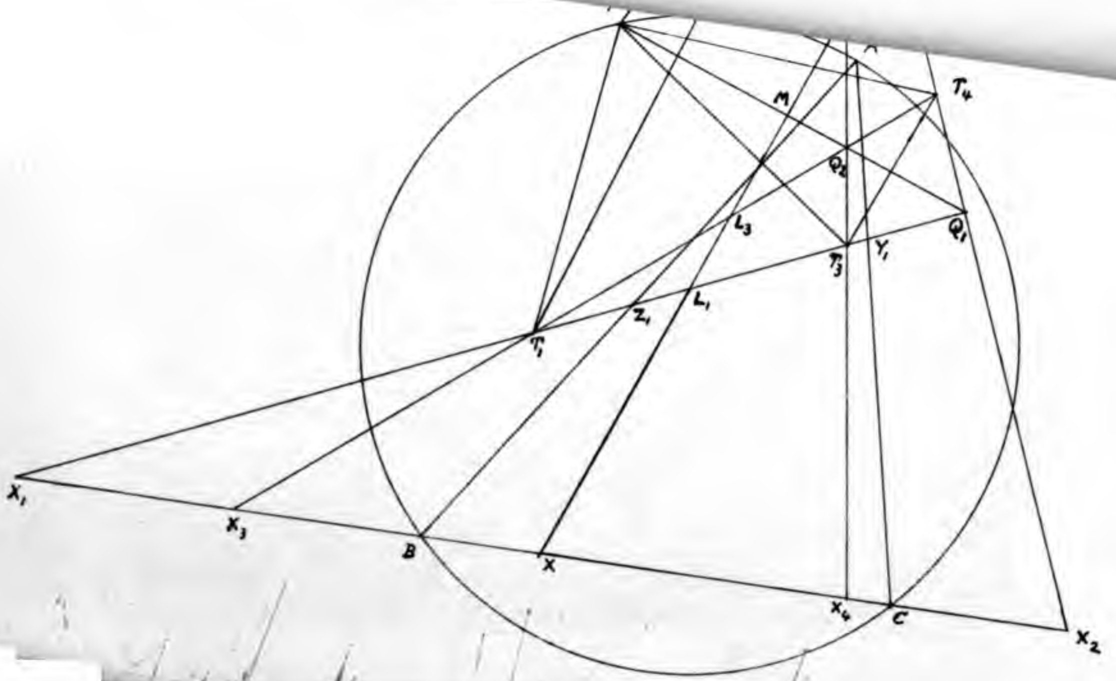

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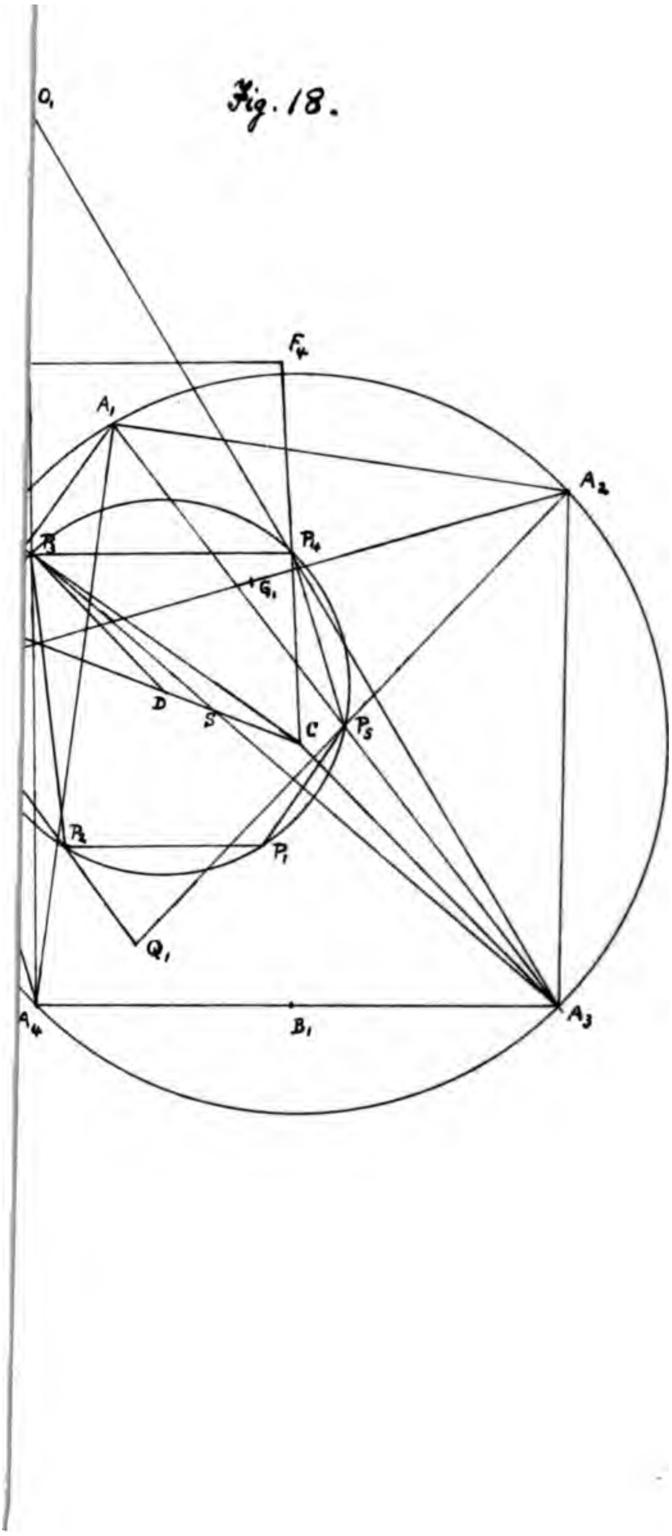
Fig. 18.



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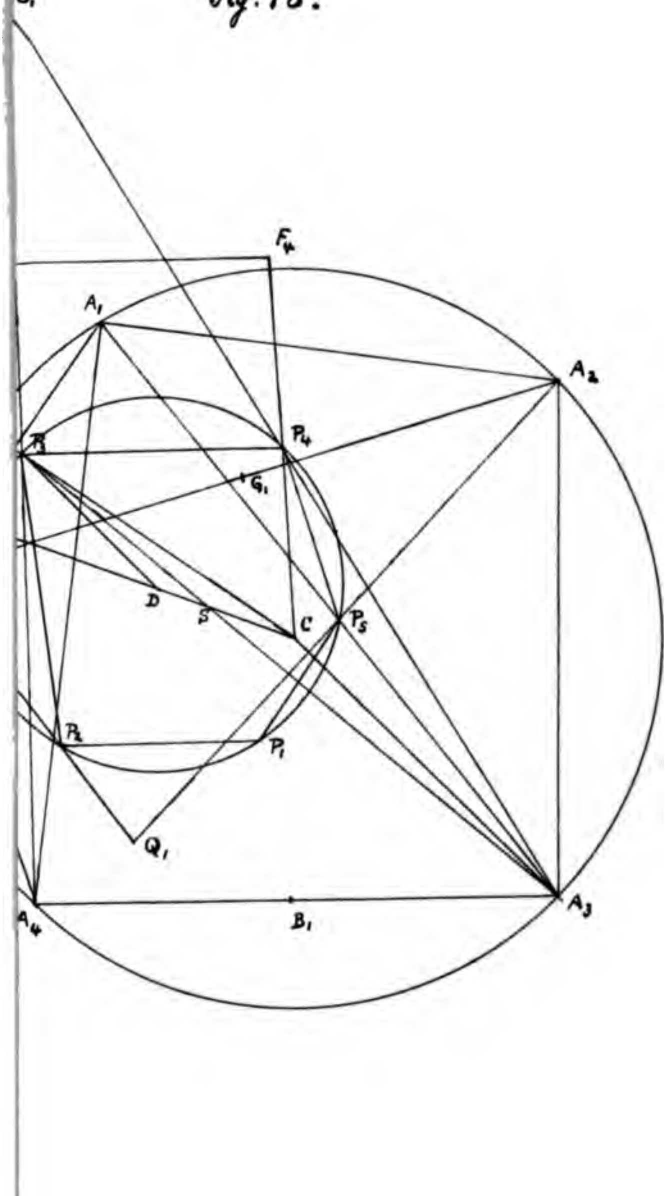

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